Sunflowers, tournaments and symmetric designs

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Fractional θ-intersecting families

Definition (Balachandran–Mathew–Mishra 2019)

Let $0 < \theta < 1$ be a rational. A family $\mathcal{F}$ of subsets of $[n]$ is called θ-intersecting if for all $A, B \in \mathcal{F}$, $A \neq B$, we have

$$|A \cap B| = \theta|A| \qquad \text{or} \qquad |A \cap B| = \theta|B|.$$

"B θ-intersects A" *"A θ-intersects B"*

Note: when $\theta = 1/2$, we also say that "A bisects B", and so on.

Question.

What is the maximum size of a fractional θ-intersecting family over $[n]$?

Example (Sunflower family)

Let $\mathcal{F}_s := \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}$.
Then, $\mathcal{F}_s$ is $\frac{1}{2}$-intersecting, and $|\mathcal{F}_s| = \frac{3n}{2} - 2$.

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Example (Hadamard family)

Let H be an $m \times m$ *Hadamard matrix* in normal form, and let J be the $m \times m$ all-ones matrix. Let $A_1, \ldots, A_{3m}$ be the rows of

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix},$$

viewed as the $\{\pm 1\}$-incidence vectors of subsets of $[2m]$.

Then, $\mathcal{F}_H := \{A_i : i \in [3m] \setminus \{1, m+1\}\}$ is a $\frac{1}{2}$-intersecting family.
Writing $2m = n$, we have $|\mathcal{F}_H| = 3n/2 - 2$.

Question.

Are these families extremal?

Even a linear upper bound is not known!

Theorem (Balachandran–Mathew–Mishra 2019)

Let $\theta \in (0,1) \cap \mathbb{Q}$. If $\mathcal{F}$ is a θ-intersecting family over $[n]$, then

$$|\mathcal{F}| \leq O_\theta(n \log(n)).$$

Conjecture (Balachandran–Mathew–Mishra 2019)

Let $\theta \in (0,1) \cap \mathbb{Q}$. There is a constant $c > 0$ such that any θ-intersecting family over $[n]$ has size at most cn.

Sunflowers, tournaments and symmetric designs

A closer look at the two examples

These two examples are at the extreme ends of a tower of *hierarchically* $\frac{1}{2}$-intersecting families.

- In the sunflower family: for any $r \geq 2$ and any sets $A_1, \ldots, A_r \in \mathcal{F}_s$ we also have

$$|A_1 \cap \cdots \cap A_r| \in \left\{ \tfrac{1}{2}|A_1|, \ldots, \tfrac{1}{2}|A_r| \right\}.$$

- In the Hadamard family: there is no $r > 2$ for which this property is satisfied.

Hierarchically r-closed fractional θ-intersecting families

Definition

Let $r \geq 2$ and $\theta \in (0,1) \cap \mathbb{Q}$. A family $\mathcal{F}$ of subsets of $[n]$ is called hierarchically r-closed θ-intersecting if, for each $2 \leq t \leq r$ and any t distinct sets $A_1, \ldots, A_t$ in $\mathcal{F}$ we have

$$\left| \bigcap_{i=1}^{t} A_i \right| \in \{\theta |A_i| : 1 \leq i \leq t\}.$$

- If a θ-intersecting family is r-closed, then it is also s-closed for all $2 \leq s \leq r$.
- If $r = 2$, then this reduces to the earlier notion of fractional θ-intersecting families.

Sunflowers . . .

Definition

A nonempty collection $\mathcal{F}$ of subsets of $[n]$ is called a sunflower if there is a set C such that $A \cap B = C$ for all distinct $A, B \in \mathcal{F}$.

- The set C is called the core of the sunflower, and the sets of the form $A \setminus C$ where $A \in \mathcal{F}$ are called petals.
- The singleton sunflower $\mathcal{F} = \{A\}$ is said to be trivial.

Example

- $\{12, 13, \ldots, 1n\}$
- $\{1234, 1256, \ldots, 12(n-1)n\}$

Sunflowers and hierarchically closed families

$\mathcal{F}_s$ is a prototypical hierarchically closed family.

In fact, if $\mathcal{F}$ is any hierarchically closed family, then

- each nonempty subfamily $\mathcal{F}(i)$ of i-sized sets is a sunflower;

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- the cores of the nontrivial sunflowers form an increasing sequence;
- *almost* all petals and cores are disjoint from each other, with at most one exception: look at the set 12 in $\mathcal{F}_s = \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}$.

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Call a θ-intersecting family with these properties to be a bouquet of sunflowers.

Question.

Does any bouquet over $[n]$ have size linear in n?

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Answer.

Yes!

Hierarchically closed families are linear in size

Theorem
(Balachandran–Bhattacharya–Kher–Mathew–S. 2023)

There is a constant $c_\theta \leq \frac{3}{2}$ *such that, if* $\mathcal{F}$ *is an* r*-closed* θ*-intersecting family over* $[n]$ *with* $r \geq 3$*, then* $|\mathcal{F}| \leq c_\theta n$*.*

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$$|\mathcal{F}| \le \lfloor \frac{3n}{2} \rfloor - 2 \qquad (*)$$

for all $n \ge 2$*.*

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When $\theta = 1/2$*,*

$$|\mathcal{F}| \leq \lfloor \tfrac{3n}{2} \rfloor - 2 \qquad (*)$$

for all $n \geq 2$*. Moreover:*

1. *(Tightness) The bound in* $(*)$ *is attained by* $\mathcal{F}_s$*.*
2. *(Uniqueness) For any family* $\mathcal{F}$ *over* $[n]$ *that attains the bound in* $(*)$*, there is a permutation* σ *of* $[n]$ *such that* $\mathcal{F}_s = \sigma(\mathcal{F}) := \{\sigma(A) : A \in \mathcal{F}\}$*.*
3. *(Stability) There exists an absolute constant* $C > 0$ *such that the following holds. If* $|\mathcal{F}| \geq (\frac{3}{2} - \epsilon)n$ *for some* $0 < \epsilon < 0.1$*, then for some permutation* σ *of* $[n]$*,* $|\sigma(\mathcal{F}) \setminus \mathcal{F}_s| < C\epsilon n$*.*

Another look at the two examples

Recall the sunflower and Hadamard families, $\mathcal{F}_s$ and $\mathcal{F}_H$.

- The sunflower family: has sets of sizes 2 and 4.
- The Hadamard family: has sets of sizes $n/2$ and $n/4$.

Question.

If the sets in a θ-intersecting family $\mathcal{F}$ are not very large, then is the size of $\mathcal{F}$ linear in n?

Bounded θ-intersecting families . . .

- Say that a family $\mathcal{F}$ is w-bounded if all the sets in $\mathcal{F}$ have size at most w.
- We now know that *bouquets* are linear in size.
- So, if we can throw away a small number of sets from a w-bounded family $\mathcal{F}$ to get a bouquet, then we will obtain a linear bound.
- Here w cannot be large, because of the Hadamard family $\mathcal{F}_H$.

Bounded θ-intersecting families . . .

Theorem (Deza 1974)

Let $\mathcal{F}$ be a w-bounded family of subsets of $[n]$ such that all pairwise intersections have the same cardinality. If $|\mathcal{F}| \geq w^2 - w + 2$, then $\mathcal{F}$ is a sunflower.

Proposition (Balachandran–Das–S. 2024)

Let $\mathcal{F}$ be a w-bounded θ-intersecting family over $[n]$. Then, there is a bouquet $\mathcal{B}$ in $\mathcal{F}$ such that $|\mathcal{F} \setminus \mathcal{B}| \leq w^3$.

Bounded θ-intersecting families are linear in size!

Theorem (Balachandran–Das–S. 2024)

Let $\theta \in (0,1) \cap \mathbb{Q}$. Let $w \leq O(n^{1/3})$ be a positive real. There is a constant $C > 0$ such that the following holds: for all sufficiently large n, if $\mathcal{F}$ is a w-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| \leq Cn$.

Bounded θ-intersecting families are linear in size!

Theorem (Balachandran–Das–S. 2024)

Let $\theta = a/b \in (0,1) \cap \mathbb{Q}$, $\gcd(a,b) = 1$. If $\mathcal{F}$ is a $o(n^{1/3})$-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| \leq (C_\theta + o(1))n$, where $C_\theta = \frac{1}{b-a} \sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$. The constant is tight for $\theta \in \{1/3\} \cup [1/2, 1)$.

Sunflowers, tournaments and symmetric designs

A linear algebraic reformulation of the original problem

Balachandran–Mathew–Mishra (2019)

Let $\mathcal{F}$ be a θ-intersecting family on $[n]$ of size m.
Let $X_{m\times n}$ be the incidence matrix for $\mathcal{F}$, so

$$X(A,x) := \begin{cases} 1, & x \in A; \\ -1, & x \notin A. \end{cases}$$

Then,

$$XX^T(A,A) = n,$$

$$XX^T(A,B) = \begin{cases} n - 2|A| + 2(1-2\theta)|B|, & \text{if } |A\cap B| = \theta|B|; \\ n - 2|B| + 2(1-2\theta)|A|, & \text{if } |A\cap B| = \theta|A|. \end{cases}$$

A linear algebraic reformulation of the original problem

Consider

$$M = \frac{1}{2}(nJ - XX^T).$$

- The diagonal of M is zero.
- The (A, B)th entry is either $f(|A|, |B|)$ or $f(|B|, |A|)$, where $f : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is given by $f(x, y) = x + (1 - 2\theta)y$.
- If $\text{rank}(M) \geq cm$, then $|\mathcal{F}| \leq \frac{1}{c}(n + 1)$.

A linear algebraic reformulation of the original problem

Let $\mathbb{F}$ be a field, $a_1, \ldots, a_n \in \mathbb{F}$, and $f : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ be a function. Denote by $\mathrm{Sym}(f; a_1, \ldots, a_n)$ the collection of all the $n \times n$ symmetric matrices over $\mathbb{F}$ with zero diagonal whose (i, j)th entries are either $f(a_i, a_j)$ or $f(a_j, a_i)$, for all $i < j$.

Question.

Is there an absolute constant $c > 0$ such that $\text{rank}(M) \geq cn$ for all $M \in \text{Sym}(f; a_1, \ldots, a_n)$?

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Answer.

No!

A matrix of rank at most 3

Consider the matrix

$$M_n = \begin{bmatrix} 0 & 1 & 4 & \cdots & (n-1)^2 \\ 1 & 0 & 1 & \cdots & (n-2)^2 \\ 4 & 1 & 0 & \cdots & (n-3)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)^2 & (n-2)^2 & (n-3)^2 & \cdots & 0 \end{bmatrix},$$

whose (i,j)th entry is $(i-j)^2$. Then $\mathrm{rank}(M_n) \leq 3$ for all $n \geq 3$, since it is the sum of three rank-one matrices.

Good pairs $(f, \underline{a})$

Definition

Let $\mathbb{F}$ be a field, $\underline{a}$ be a sequence in $\mathbb{F}$, and $f : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ be a function. We say that $(f; \underline{a})$ is a good pair over $\mathbb{F}$ if $f(a_i, a_i) \neq 0$ for all i.

Good pairs $(f, \underline{a})$

Example

Let $\theta \in \mathbb{F}$. Define $f_\theta \colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ to be the function

$$f_\theta(x, y) := x + (1 - 2\theta)y$$

for all $x, y \in \mathbb{F}$. Then, $(f_\theta, \underline{a})$ is a good pair over $\mathbb{F}$ if $f_\theta(a_i, a_i) = 2(1 - \theta)a_i$ is nonzero for all i; that is, if:

- $\mathrm{char}(F) \neq 2$,
- $a_i \neq 0$ for all i,
- $\theta \neq 1$.

Question (refined).

Let $(f; \underline{a})$ be a good pair over $\mathbb{F}$.
Is there an absolute constant $c > 0$ such that $\mathrm{rank}(M) \geq cn$ for all $M \in \mathrm{Sym}(f; a_1, \ldots, a_n)$?

Question (refined).

Let $(f;\underline{a})$ be a good pair over $\mathbb{F}$.
Is there an absolute constant $c > 0$ such that $\operatorname{rank}(M) \geq cn$ for all $M \in \operatorname{Sym}(f; a_1, \ldots, a_n)$?

Answer.

Yes, with high probability!

A connection with tournaments

Definition

- A tournament T over $[n]$ is an orientation of the edges of the complete graph K_n.
- T is transitive if there is an ordering $\prec$ on $[n]$ such that $i \to j \iff i \prec j$.
- If $\prec \equiv <$, then we shall say that T is in the natural ordering.

A connection with tournaments

For a tournament T over $[n]$, define the associated matrix $M_T \in \mathrm{Sym}(f; a_1, \ldots, a_n)$ by

$$M_T(i,j) := \begin{cases} 0, & i = j; \\ f(a_i, a_j), & i < j,\ i \to j \text{ in } T; \\ f(a_j, a_i), & i < j,\ j \to i \text{ in } T. \end{cases}$$

Conversely, given a matrix $M \in \mathrm{Sym}(f; a_1, \ldots, a_n)$, we can define a tournament over $[n]$ (in possibly more than one way).

Matrices associated to *random* tournaments have high rank

Theorem (Balachandran–Bhattacharya–S. 2023)

Let $(f_\theta, \underline{a})$ be a good pair over $\mathbb{F}$. For a uniformly random tournament T over $[n]$, with high probability we have

$$\mathrm{rank}(M_T) \geq \left(\frac{1}{2} - o(1)\right) n.$$

Main idea

Theorem (McDiarmid 1989)

Let $X_1, \ldots, X_n$ be independent random variables, with X_k taking values in a set Ω_k for each k. Suppose that the measurable function $f : \prod_k \Omega_k \to \mathbb{R}$ satisfies, for each k,

$$|f(\underline{x}) - f(\underline{x}')| \leq c_k$$

whenever the vectors $\underline{x}$ and $\underline{x}'$ differ only in the kth coordinate. Let Y be the random variable $f(X_1, \ldots, X_n)$. Then, for any $t > 0$,

$$\mathbb{P}(|Y - \mathbb{E}(Y)| > t) \leq 2\exp\left(-2t^2 / \sum_k c_k^2\right).$$

Matrices associated to *transitive* tournaments have almost full rank

Explicit examples of matrices in $\mathrm{Sym}(f_\theta; a_1, \ldots, a_n)$ with high rank can be found when $\theta = 1/2$ and $M = M_T$ for a transitive tournament T.

Matrices associated to *transitive* tournaments have almost full rank

Let $\mathbb{F}$ have arbitrary characteristic, let T be a tournament over $[n]$, and let $M_T(\underline{a}) \in \mathrm{Sym}(f_{1/2}; a_1, \ldots, a_n)$.

Theorem (Balachandran–Bhattacharya–S. 2024)

Let T_n be the transitive tournament over $[n]$ in the natural orientation. For all $n \geq 2$, $\det(M_{T_n}(\underline{a}))$ satisfies

$$\det(M_{T_n}(\underline{a})) = -a_{n-1}^2 \det(M_{T_{n-2}}(\underline{a})) - 2a_{n-1} \det(M_{T_{n-1}}(\underline{a})).$$

Matrices associated to *transitive* tournaments have almost full rank

Corollary (Balachandran–Bhattacharya–S. 2024)

If T is a transitive tournament on $[n]$, then

$$\mathrm{rank}(M_T(\underline{a})) \geq n - 1.$$

Sunflowers, tournaments and symmetric designs

Low-rank matrices in Sym$(f; a_1, \ldots, a_n)$

- The best-known low-rank matrices are those arising from the maximal θ-intersecting families $\mathcal{F}_s$ and $\mathcal{F}_H$, and these have rank $\frac{2n}{3}$.
- Let $\underline{a}$ take only two distinct values, say α and β.
- Let $(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$ be denoted by $(\alpha^{(m)}, \beta^{(n)})$.

Question.

Let $(f; \alpha^{(m)}, \beta^{(n)})$ be a good pair over $\mathbb{F}$.
How low can the rank of a matrix
$M \in \mathrm{Sym}(f; \alpha^{(m)}, \beta^{(n)})$ be?

Bipartite graphs and matrices in $\mathrm{Sym}(f;\alpha^{(m)},\beta^{(n)})$

Note that any element in $\mathrm{Sym}(f;\alpha^{(m)},\beta^{(n)})$ has the following block matrix form:

$$\begin{bmatrix} f(\alpha,\alpha)(J_m - I_m) & C \\ C^T & f(\beta,\beta)(J_n - I_n) \end{bmatrix}.$$

Here, each entry of C is either $f(\alpha,\beta)$ or $f(\beta,\alpha)$.
Let G_M be the bipartite graph defined by the $f(\alpha,\beta)$ entries.

Bipartite graphs and matrices in $\mathrm{Sym}(f; \alpha^{(m)}, \beta^{(n)})$

Multiplicity of eigenvalues vs. rank

Theorem (Balachandran–S. 2024)

Let $M \in \mathrm{Sym}(f; \alpha^{(m)}, \beta^{(n)})$, where $f(\alpha, \alpha)$, $f(\beta, \beta)$, and $f(\alpha, \beta) - f(\beta, \alpha)$ are all nonzero. Let $\mu(f; \alpha, \beta) \in \mathbb{C}$ such that

$$\big(\mu(f; \alpha, \beta)\big)^2 = \frac{f(\alpha, \alpha) f(\beta, \beta)}{(f(\alpha, \beta) - f(\beta, \alpha))^2}.$$

If ν is the multiplicity of $\mu(f; \alpha, \beta)$ as an eigenvalue of G_M, then

$$m + n - \nu - 2 \leq \mathrm{rank}(M) \leq m + n - \nu + 2.$$

In particular, if $f(\alpha, \alpha) f(\beta, \beta) < 0$, then $\mathrm{rank}(M) \geq m + n - 2$.

Symmetric designs

Theorem (Rowlinson 2016)

Let G be a connected bipartite graph of order $n > 5$, with $\mu \notin \{-1, 0\}$ as an eigenvalue of multiplicity $\nu > 1$.

(a) *If d is the maximum degree in G, then $\nu \leq n - 1 - d$.*

(b) *If equality holds in (a), then $\nu \leq d - 1$.*

(c) *If equality holds in (b), then G is the bipolar cone over a graph G_0, where G_0 is either the incidence graph of a symmetric 2-design, or a 2-balanced bipartite graph.*

Symmetric designs

Definition

A symmetric 2-(v, k, λ) design is a collection of k-subsets of $[v]$ such that every pair of elements in v belongs to exactly λ sets in the collection.

- Any symmetric 2-(v, k, λ) design has an associated bipartite point-block incidence graph, which has spectrum

$$\{(v)^{(1)}, (\sqrt{k-\lambda})^{(v-1)}, (-\sqrt{k-\lambda})^{(v-1)}, (-v)^{(1)}\}.$$

Low-rank *real* matrices in $\mathrm{Sym}(f_\theta; \alpha^{(m)}, \beta^n)$

Theorem (Balachandran–S. 2024)

Let $\theta \in (0, 1)$. Let $\beta \in \mathbb{R}$ be a root of the quadratic equation

$$x^2 - (2 + (\theta^{-1} - 1)^2)x + 1 = 0.$$

For every $n \in \mathbb{N}$, there is a matrix $M \in \mathrm{Sym}(f_\theta; 1^{(n)}, \beta^{(n)})$ with $\mathrm{rank}(M) \leq n + 3$.

These matrices are constructed from the complete bipartite graph $K_{n,n}$ minus a perfect matching.

Low-rank *rational* matrices in $\mathrm{Sym}(f_{1/2}; \alpha^{(m)}, \beta^{(n)})$

Theorem (Balachandran–S. 2024)
For each $\varepsilon > 0$, there exists $c_\varepsilon \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ and $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{Q}$ such that there is a sequence of matrices $M_n \in \mathrm{Sym}(f_{1/2}; \alpha_\varepsilon^{(n)}, \beta_\varepsilon^{(n)})$ for which $\mathrm{rank}(M_n) \le c_\varepsilon n + O(1)$.

These matrices are constructed from Hadamard designs, which are symmetric 2-$(4n-1, 2n-1, n-1)$ designs.

Ruling out candidates for low-rank matrices

Proposition (Balachandran–S. 2024)

Let Δ *be a symmetric* 2-(v, k, λ) *design with* $k - \lambda = p^m$ *for some prime* p *and integer* $m \geq 1$. *Consider* $M_\Delta \in \mathrm{Sym}(f_{1/2}; \alpha^{(v)}, \beta^{(v)})$. *If* $\mathrm{rank}(M_\Delta) \leq v + 3$, *then* $p^m = 2$.

This rules out most of the known examples of symmetric designs as viable candidates for inducing matrices of low rank.

For instance, none of Families 1, 7, 8, 9, 13, or 14 from the *Handbook of Combinatorial Designs* are viable, except for the Fano plane $PG(2, 2)$ from Family 1.

The Fano family

If we demand $\alpha : \beta :: 1 : 2$, then the Hadamard construction is longer helpful.

In fact, the 2-$(7,3,1)$ design (i.e., the Fano plane) and the 2-$(7,4,2)$ design are the only ones that produces matrices of low-rank and with entries in the proportion $1:2$ [Royle 2023].

This construction gives matrices $M_n \in \mathrm{Sym}(f_{1/2}; 1^{(7n)}, 2^{(7n)})$ of rank at most $\frac{4}{7}n$.

Furthermore, for $n = 1$, there is an associated $\frac{1}{2}$-intersecting family $\mathcal{F}_{\mathrm{Fano}}$ over $[8]$ of size 14:

$$\mathcal{F}_{\mathrm{Fano}} = \mathcal{F}_s \cup \{1357, 1368, 1458, 1467\}.$$

What next?

Open questions

- Suppose $\mathcal{F}$ is a hierarchically r-closed $\frac{a}{b}$-intersecting family over $[n]$ such that $|A| \geq kb$ for all $A \in \mathcal{F}$. What can be said about $|\mathcal{F}|$?
- Suppose $\mathcal{F}$ is a hierarchically r-closed $\frac{a}{b}$-intersecting family $[n]$ such that no two sets have the same size. Is it true that $|\mathcal{F}| \leq O(n)$?
- For what kind of functions $f : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ other than f_θ can we prove a linear bound on the ranks of matrices in $\mathrm{Sym}(f; a_1, \ldots, a_n)$?
- Consider analogous families like $\mathrm{SkewSym}(f; a_1, \ldots, a_n)$. Matrices associated to transitive tournaments have high rank here, too. What about a matrix associated to a uniformly random tournament?
- Can the $\frac{4}{7}n$ bound on the rank obtained from the Fano construction be improved? That is, is there a bipartite graph with $\sqrt{2}$ as an eigenvalue with "high multiplicity", specifically more than in the Heawood graph?

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