Some problems in combinatorics:

Excursions in graph colorings and extremal set theory

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by

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Dedicated to all my teachers

Thesis Approval

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Abstract

This thesis has two parts. Part I is concerned with graph coloring problems, and Part II with problems arising in extremal set theory.

In Part I, our focus is on the colorability and list colorability of graphs. This part is divided into five chapters, including the introduction (Chapter 1).

- In Chapter 2, we give a detailed proof of a theorem due to Altshuler: every 6-regular triangulation of the torus is isomorphic to a "standard" 6-regular triangulation $T(r, s, t)$, where $r, s \geq 1$, $0 \leq t \leq s - 1$. These triangulations will appear repeatedly in the rest of the chapters in Part I.

- In Chapter 3, we introduce the following parameter to study the gap between the chromatic number $\chi(G)$ and list chromatic number $\chi_\ell(G)$ for any loopless graph G. Define $\mathsf{jump}(g) := \max\{\chi_\ell(G) - \chi(G) : G \text{ is loopless and embeddable in } S_g\}$ for $g \geq 0$, where S_g is the orientable surface of genus g. It follows from well-known results in the literature that $\mathsf{jump}(0) = 2$. We investigate $\mathsf{jump}(1)$, and show that it is bounded above by 2 for any 6-regular triangulation of the torus. We also show that the largest gap for graphs embeddable in S_g, i.e. $\mathsf{jump}(g)$, is of the order $\Theta(\sqrt{g})$, and moreover for graphs with chromatic number of the order $o(\sqrt{g}/\log_2(g))$ the largest gap is of the order $o(\sqrt{g})$. This chapter is based on the work published in [BS21].

- In Chapter 4, we explicitly describe a linear time algorithm for 5-list coloring a large class of 6-regular triangulations of the torus. We also show that none of these graphs are 3-choosable, so that their list chromatic numbers are either 4 or 5. This chapter is based on the work published in [BS23].

- In Chapter 5, we investigate the colorability of the 6-regular triangulations of the torus. In particular, a gap in the literature regarding the 4-colorability of these graphs is identified and fixed. This completes the characterization of the colorability of these triangulations. This chapter is based on the work published in [San22].

In Part II, our focus is on a fractional variant of intersecting families of sets. This part is divided into three chapters, including the introduction (Chapter 6).

- In Chapter 7, we consider a hierarchical version of fractional θ-intersecting families, $\theta \in (0,1) \cap \mathbb{Q}$. We exhibit a linear upper bound on the size of any such hierarchically closed family. When $\theta = 1/2$, so that the family is "bisection closed", we prove that the linear upper bound is tight as well as stable. This chapter is based on the work published in [BBK+23].

- In Chapter 8, we compute the ranks of certain families of matrices that naturally arise from fractional θ-intersecting families. We show that these matrices—which can be seen to correspond to tournaments—all have "high" rank with high probability. We also show that a certain class of such matrices—which correspond to transitive tournaments—are of "almost" full rank. This chapter is based on the work published in [BBS23, BBS24].

Keywords: chromatic number, list chromatic number, linear time algorithm, triangulation, fractional intersecting family, sunflower, generalized adjacency matrix, and McDiarmid's inequality.

Contents

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Abbreviations

iff	if and only if
whp	with high probability

Symbols

$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, \ldots\}$
$\mathbb{Z}_{>0}$	the set of positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$[n]$	the set $\{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$
$\mathbb{F}$	a field
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$:=$	is defined as
$=:$	is defined by
$\subset$, $\subseteq$	subset
$\subsetneq$	proper subset
$O(\,\cdot\,)$	big omicron; $f \leq O(g)$ if $\exists\, M > 0$ such that $\limsup_{x\to\infty} f(x)/g(x) \leq M$
$o(\,\cdot\,)$	little omicron; $f = o(g)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$
$\Omega(\,\cdot\,)$	big omega; $f \geq \Omega(g)$ if $g \leq O(f)$
$\omega(\,\cdot\,)$	little omega; $f = \omega(g)$ if $g = o(f)$
$\Theta(\,\cdot\,)$	big theta; $f = \Theta(g)$ if $\Omega(g) \leq f \leq O(g)$
v	number of vertices, $\lvert V\rvert$
e	number of edges, $\lvert E\rvert$
f	number of faces, $\lvert F\rvert$
χ	chromatic number
χ_ℓ	list chromatic number
$\binom{X}{i}$	the collection of all subsets of X of size i

Part I

Graph colorings

Chapter 1

Introduction

1.1 Preliminaries

We shall denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$. For each $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. We freely make use of the Bachmann–Landau–Knuth notations, but state them briefly for completeness below. Let f and g be positive functions of real variables.

1. $f = o(g)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$;
2. $f = \omega(g)$ if $g = o(f)$;
3. $f \leq O(g)$ if there is a constant $M > 0$ such that $\limsup_{x\to\infty} f(x)/g(x) \leq M$;
4. $f \geq \Omega(g)$ if $g \leq O(f)$; and
5. $f = \Theta(g)$ if $\Omega(g) \leq f \leq O(g)$.

For integers $r \geq 1$, $s \geq 1$ and $0 \leq t \leq s-1$, take $V = \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ to be the vertex set of the graph $T(r, s, t)$ equipped with the following edges:

- For each $1 < i < r$, (i, j) is adjacent to $(i, j \pm 1)$, $(i \pm 1, j)$ and $(i \pm 1, j \mp 1)$.
- If $r > 1$, $(1, j)$ is adjacent to $(1, j \pm 1)$, $(2, j)$, $(2, j-1)$, $(r, j+t+1)$ and $(r, j+t)$.
- If $r > 1$, (r, j) is adjacent to $(r, j \pm 1)$, $(r-1, j+1)$, $(r-1, j)$, $(1, j-t)$ and $(1, j-t-1)$.
- If $r = 1$, $(1, j)$ is adjacent to $(1, j \pm 1)$, $(1, j \pm t)$ and $(1, j \pm (t+1))$.

Here, addition in the first coordinate is taken modulo r and in the second coordinate is taken modulo s. Figure 1.1 depicts the graph $G = T(5, 6, 2)$; note that the edges between the top and bottom rows are not shown.

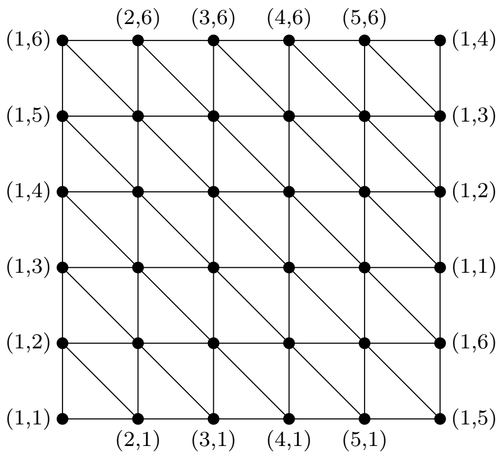


Figure 1.1: $G = T(5, 6, 2)$. The edges between the vertices in the top and bottoms rows are not shown.

It is clear that each $T(r, s, t)$ embeds in the torus as a triangulation, i.e., it has a drawing on the torus without any crossing edges and such that each face in the drawing is bounded by three edges (this is described in full detail in Chapter 2). Altshuler's theorem says that these are all the 6-regular triangulations on the torus up to isomorphism.

Theorem 1.1 (Altshuler [11, 12]). *Every 6-regular triangulation on the torus is isomorphic to $T(r, s, t)$ for some integers $r \geq 1$, $s \geq 1$, and $0 \leq t < s$.*

To clarify, an isomorphism of embeddings is a bijection of the vertex sets that maps corresponding edges to edges as well as faces to faces. In Chapter 2 of this thesis, we provide a detailed proof of Altshuler's theorem.

A *(vertex) coloring* of a graph $G = (V, E)$ is an assignment of a "color" to each vertex; more formally, it is a function $v \mapsto \mathsf{color}(v) \in \mathbb{N}$ for every $v \in V(G)$. A coloring of G is *proper* if adjacent vertices receive distinct colors. The graph G is *k-colorable* if there exists a proper coloring of the vertices using k colors. The least integer k for which G is k-colorable is called the *chromatic number* of G and is denoted $\chi(G)$. If $\chi(G) = k$, we also say that G is *k-chromatic*.

A *list assignment* $\mathcal{L}$ on G is a collection of sets of the form $\mathcal{L} = \{L_v \subset \mathbb{N} : v \in V(G)\}$, where one thinks of each L_v as a *list* of colors available for coloring the vertex $v \in V(G)$. The graph G is *$\mathcal{L}$-choosable* if there exists a proper coloring of the vertices such that $\mathsf{color}(v) \in L_v$ for every $v \in V(G)$. A *k-list* is a list of size greater than or equal to k, and a *k-list assignment* is an assignment of k-lists. The graph G is *k-choosable* if it is $\mathcal{L}$-choosable for every k-list assignment $\mathcal{L}$. The least integer k for which G is k-choosable is called the *choice number* or *list chromatic number* of G and is denoted $\chi_\ell(G)$. If $\chi_\ell(G) = k$, we also say that G is *k-list chromatic*.

1.2 The gap between the chromatic and list chromatic numbers for graphs embeddable on surfaces

For a given graph G, how far apart can its chromatic and list chromatic numbers be? In Chapter 3 of the thesis, we formulate this question carefully, and provide explicit bounds for a large class of graphs embeddable on the torus, and asymptotic bounds for graphs embeddable on a surface of genus $g \geq 0$. We also provide asymptotics that show that the maximum gap cannot be achieved by graphs of "small" chromatic number.

1.2.1 Motivation

The choice number was defined independently by Vizing [99] and Erdős, Rubin and Taylor [40]. It generalizes the chromatic number in the following sense: a graph G is k-colorable if and only if it is $\mathcal{L}$-choosable for any k-list assignment in which all the lists L_v are identical. Hence, $\chi(G) \leq \chi_\ell(G)$ for any graph G. In general, however, there exist graphs G for which $\chi(G) < \chi_\ell(G)$ (i.e., for which strict inequality holds), so it behooves one to investigate the nature of the gap between the choice number and chromatic number.

One line of investigation is to examine *chromatic-choosable* graphs, that is, graphs that satisfy $\chi(G) = \chi_\ell(G)$. We mention one of the important results concerning such graphs, conjectured by Ohba [78] and subsequently settled in the affirmative by Noel, Reed and Wu [77]: if G is a graph on at most $2\,\chi(G)+1$ vertices, then it satisfies $\chi(G) = \chi_\ell(G)$.

The opposite line of investigation is to examine the width of the gap between the chromatic number and choice number. In the papers by Vizing and Erdős–Rubin–Taylor, it is shown that there exist bipartite graphs (that is, graphs with $\chi(G) = 2$) that have arbitrarily large choice number; more precisely, they showed that $\chi_\ell(K_{n,n}) > k$ if $n \geq \binom{2k-1}{k}$. At first glance it appears that this line of investigation is thus fruitless, but one has to note that the graphs $K_{n,n}$ have high average degree. In fact, Alon [6] showed that $\chi_\ell(G) \geq \left(\frac{1}{2} - o(1)\right) \log(\delta)$, where $\delta \equiv \delta(G)$ is the minimum degree of G.

Thus, one is motivated to bound the minimum degree of graphs in order to examine the gap between the chromatic number and choice number. A natural criterion for doing so is to consider graphs that are embeddable in a fixed surface, as we show below. By a *surface* we mean a compact connected real 2-manifold without boundary. The classification of surfaces theorem says that every orientable surface is homeomorphic to a sphere with $g \geq 0$ handles attached, denoted S_g, and every nonorientable surface is homeomorphic to a sphere with $k \geq 1$ crosscaps attached, denoted N_k. The *genus* of the surface S_g (resp.

N_k) is defined to be g (resp. k). For instance, S_0 is the sphere, S_1 is the torus, N_1 is the projective plane, and N_2 is the Klein bottle.

Informally, a graph $G = (V, E)$ is *embeddable* in a surface Σ if there exists a drawing (or *embedding*) of G on the surface without any crossing edges. A *face* of an embedding $\sigma\colon G \to \Sigma$ is a connected component of $\Sigma - \sigma(G)$. If every face is homeomorphic to a disc, then we call σ a 2-*cell embedding*. For a face f of a 2-cell embedding, $\mathsf{degree}(f)$ is the number of edges in its *boundary circuit* (or *facial circuit*). A 2-cell embedding is a *triangulation* if every face is bounded by three edges, that is, if every face has degree equal to 3. We shall primarily restrict our attention to 2-cell embeddings of graphs on orientable surfaces.

Now, suppose that $G = (V, E)$ is a connected *simple* graph (i.e., without multi-edges or loops) with at least 3 vertices, and let S_g be the surface of minimal genus $g \geq 0$ for which there exists an embedding of G into S_g. Choose an embedding into S_g and denote by v, e, and f the number of vertices, edges, and faces, respectively, of G in this embedding. A theorem of Youngs [105] says that this must be a 2-cell embedding, so by the Euler–Poincaré formula we have that $\mathsf{v} - \mathsf{e} + \mathsf{f} = 2 - 2g$. Since $\mathsf{degree}(f) \geq 3$ for every face f, we have $3\mathsf{f} \leq 2\mathsf{e}$, and this bounds the minimum degree as $\delta(G) \leq 2\mathsf{e}/\mathsf{v} \leq 6 + 12(g-1)/\mathsf{v}$. Hence, $\delta(G) \leq 5$ if $g = 0$ and $\delta(G) \leq 4g + 2$ if $g \geq 1$.

Thus, we make the following definition:

Definition 1.2. For a loopless graph $G = (V, E)$, define the *jump of* G by $\mathsf{jump}(G) := \chi_\ell(G) - \chi(G)$. For each $g \geq 0$, define the *jump at* g by

$$\mathsf{jump}(g) := \max\{\mathsf{jump}(G) : G \text{ is loopless and embeddable in } S_g\}.$$

For graphs embeddable in the sphere S_0 (equivalently, for planar graphs), the investigation of $\mathsf{jump}(0)$ was indicated by Erdős–Rubin–Taylor [40] through the following conjectures and question:

(C1) Every planar graph is 5-choosable.

(C2) There exists a planar graph that is not 4-choosable.

(Q) Does there exist a planar bipartite graph that is not 3-choosable?

Alon and Tarsi [10] answered **(Q)** in the negative by showing that every planar bipartite graph is 3-choosable; this is also best possible since there are simple examples [40] of planar bipartite graphs that are not 2-choosable. Voigt [100] settled **(C2)** positively by constructing a planar graph on 238 vertices that is not 4-choosable, and Thomassen [97] settled **(C1)** positively through a remarkably short and elegant proof.

Thus, $\mathsf{jump}(0) \leq 2$, with equality holding if and only if there exists a planar 3-chromatic graph that is not 4-choosable. Mirzakhani [70] constructed such a graph on 63 vertices; independently, Voigt and Wirth [101] observed that a non-4-choosable planar graph on 75 vertices constructed by Gutner [51] is 3-chromatic. Thus, it is known that $\mathsf{jump}(0) = 2$.

1.2.2 Our work

We pose the analogous question for toroidal graphs:

Question 1.3. *What is* $\mathsf{jump}(1)$*? That is, how large can the gap between the choice number and chromatic number be for a toroidal graph?*

Since every planar graph is also toroidal, the maximum gap for toroidal graphs cannot be smaller than 2. In Chapter 3 of this thesis, we examine 6-regular triangulations on the torus and show the following result:

Theorem 1.4. *For any loopless* 6*-regular triangulation* G *on the torus,* $\mathsf{jump}(G) \leq 2$.

In particular, we show that any 3-chromatic 6-regular triangulation of the torus is 5-choosable, by using an application of the combinatorial nullstellensatz due to Alon and Tarsi [10].

While computing $\mathsf{jump}(g)$ precisely for larger values of g seems difficult, we are able to describe the asymptotic behavior of $\mathsf{jump}(g)$ as follows:

Theorem 1.5. $\mathsf{jump}(g) = \Theta(\sqrt{g})$. *That is, there exist positive constants* c_1 *and* c_2 *such that*

$$c_1\sqrt{g} \leq \mathsf{jump}(g) \leq c_2\sqrt{g}$$

for all sufficiently large g.

A natural follow-up is to investigate for which graphs this largest gap is attained, for which the following classical result is useful:

Theorem 1.6 (Heawood [52], Borodin [56]). *Let* $g \geq 1$. *If the loopless graph* G *is embeddable in* S_g*, then*

$$\chi(G) \leq \chi_\ell(G) \leq H(g) := \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

Heawood proved that $H(g)$, the so called *Heawood number*, is an upper bound for $\chi(G)$, and essentially the same argument carries forward to prove that $H(g)$ is an upper bound for $\chi_\ell(G)$ as well. Now, Theorem 1.6 shows that if a loopless graph G that is embeddable in S_g is $H(g)$-chromatic, then it is also $H(g)$-list-chromatic, so such graphs can never attain the maximum gap for S_g. The same is true if G is $\big(H(g)-1\big)$-chromatic, by the following result known as Dirac's map color theorem:

Theorem 1.7 (Dirac–Ungar [35], Böhme–Mohar–Stiebitz [22])**.** *Let $g \geq 1$. If G is embeddable in S_g with $\chi(G) = H(g)$ or $\chi_\ell(G) = H(g)$, then $K_{H(g)}$ is a subgraph of G.*

Dirac and Ungar proved this result for the chromatic number, and it was later extended to the choice number by Böhme, Mohar and Stiebitz.

At the other end, consider the complete bipartite graph $K_{n,n}$. It is an easy exercise that $K_{n,n}$ is k-choosable where $k := \lfloor \log(n) \rfloor + 1$, and it is known that $K_{m,n}$ is embeddable in S_g for $g := \lceil (m-2)(n-2)/4 \rceil$ and this is best possible (see [86]). Hence,

$$\mathsf{jump}(K_{n,n}) \leq \log(n) - 1 \leq \log\big(2\sqrt{g+1}+2\big) - 1 \leq O\big(\log(g)\big) \leq o(\sqrt{g}).$$

Since any bipartite graph is a subgraph of a complete bipartite graph, this shows that one does not expect bipartite graphs to attain the greatest gap on a fixed surface.

This motivates the following definition:

Definition 1.8. For each $g \geq 0$, $r \geq 1$, define

$$\mathsf{jump}(g,r) := \max\{\mathsf{jump}(G) : G \text{ is loopless and embeddable in } S_g,\ \chi(G) = r\}.$$

Note that if there is no r-chromatic graph embeddable in S_g, then $\mathsf{jump}(g,r) = 0$. We prove the following stronger result along the same lines as the bipartite case:

Theorem 1.9. *$\mathsf{jump}(g,r) = o(\sqrt{g})$ when $r = o\big(\sqrt{g}/\log(g)\big)$. That is, if for each $\delta > 0$ we have $r \leq \delta\sqrt{g}/\log(g)$ for all sufficiently large g, then for every $\epsilon > 0$, $\mathsf{jump}(g,r) \leq \epsilon\sqrt{g}$ for all sufficiently large g.*

1.3 List coloring the regular toroidal triangulations in linear time

In our proof of Theorem 1.4 in Chapter 3, we found that any 3-chromatic 6-regular triangulation of the torus is 5-choosable. In Chapter 4, we revisit this calculation from a computational viewpoint, and provide a linear time algorithm for 5-list coloring a large class of 6-regular triangulations on the torus that includes the 3-chromatic ones that we previously considered.

1.3.1 Motivation

k-choosability is computationally hard

It is well-known that computing the chromatic number is an NP-hard problem [61]. Moreover, many restricted versions of the colorability problem are also computationally hard. For instance, finding a 4-coloring of a 3-chromatic graph is NP-hard [58]. Even the problem of 3-colorability of 4-regular planar graphs is known to be NP-complete [34].

Naturally, list coloring is also a computationally hard problem, but much more: for instance, it is well-known [51] that the problem of deciding whether a given planar graph is 4-choosable is NP-hard—even if the 4-lists are all chosen from $\{1, 2, 3, 4, 5\}$ [33]—and so is deciding whether a given planar triangle-free graph is 3-choosable [51]. But, contrast the latter with the fact that every planar triangle-free graph is 3-colorable by Grötzsch's theorem [50], and that a 3-coloring can be found in linear time [38]. In other words, restrictions on graph parameters—such as the girth, as in Grötzsch's theorem—that allow for efficient coloring algorithms need to be strengthened further in order to get list coloring algorithms of a similar flavor.

Note that even proving nontrivial bounds for the choice number is far tougher than the corresponding problem for the chromatic number. Some of the notable instances of such bounds being determined include Brooks's theorem for choosability [40, 99], Thomassen's remarkable proof that every planar graph is 5-choosable [97], and Galvin's solution to the famous Dinitz problem [47]. Other interesting examples include the fact that planar bipartite graphs are 3-choosable [10] and that any 4-regular graph decomposable into a Hamiltonian circuit and vertex-disjoint triangles is 3-choosable [43]. However, there is a fundamental difference between the former and latter examples, as we elaborate below.

$\mathcal{L}$-coloring is algorithmically hard

Consider the following problem: given a list assignment $\mathcal{L}$ on a graph G, can one efficiently determine whether or not G is $\mathcal{L}$-choosable, and in the case when G is $\mathcal{L}$-choosable can one also efficiently specify a proper coloring from these lists? The theorems of Brooks, Thomassen and Galvin mentioned earlier are some of the few instances where such algorithms are known for a large class of graphs. In the other examples that we mentioned, the proof uses the combinatorial nullstellensatz [5], in particular a powerful application found by Alon and Tarsi [10]. Hence, it does not allow one to extract an efficient algorithmic solution to the problem of $\mathcal{L}$-coloring when the list assignment $\mathcal{L}$ is specified, except in certain special cases. That there is no known efficient algorithm that produces a 3-list coloring from a given list assignment in these examples illustrates the difficulty of the problem of efficiently finding a proper $\mathcal{L}$-coloring even for graphs of small maximum degree. Even just for planar bipartite graphs, an algorithmic determination of a list coloring largely remains open [33].

Hence, efficient $\mathcal{L}$-coloring algorithms for large classes of graphs are interesting. We also place our work within the context of recent results on efficient list coloring algorithms for similar classes of graphs in Section 1.3.3 below.

1.3.2 Our work

As noted earlier, in order to find good bounds for the choice number it is natural to place restrictions on certain graph parameters. We shall focus on a certain class of graphs G having bounded *degeneracy number* $d(G)$, defined as

$$d(G) := \max\{\delta(H) : H \text{ is an induced subgraph of } G\},$$

where $\delta(H)$ is the minimum degree of H. A simple inductive argument [4] shows that $\chi_\ell(G) \leq d(G)+1$ for every simple graph G. This improves the rudimentary upper bound $\chi_\ell(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. A natural choice for a large collection of graphs with bounded degeneracy number is the class of graphs that are embeddable in a fixed surface. In Chapter 4, we will be concerned only with toroidal graphs, that is, graphs that are embeddable on the torus S_1.

Let $G = (V, E)$ be a toroidal graph, and let F be the set of its faces in a 2-cell embedding into S_1. The graphs satisfying $\mathsf{degree}(v) = d$ for all $v \in V$ and $\mathsf{degree}(f) = m$ for all $f \in F$, for some $d, m \geq 1$, have been of interest in a wide variety of contexts. A simple calculation using the Euler–Poincaré formula shows that the only possible values of (d, m) are $(3, 6)$, $(4, 4)$ and $(6, 3)$. Our focus will be on the graphs of the last kind, namely the 6-regular triangulations on the torus. Since triangulations have the maximum possible number of edges in any graph with a fixed number of vertices and embeddable on a given surface, one might additionally expect this class of graphs to present a greater obstacle to an efficient solution to the list coloring problem as compared to the others.

The main result of Chapter 4, Theorem 1.10, is a linear time algorithm for 5-list coloring a large class of these toroidal 6-regular triangulations. Our result is nearly tight for this class in the sense that the list size is at most one more than the choice number for any graph in this family. In fact, in Corollary 1.11 we find an infinite family of 5-chromatic-choosable graphs for which a list coloring can be specified in linear time.

Recall that $T(r, s, t)$ is a triangulation obtained from an $r \times s$ toroidal grid $(r, s \geq 1)$ as described in Section 1.1 above.

Theorem 1.10. *Let G be a simple 6-regular toroidal triangulation. Then, G is 5-choosable under any of the following conditions:*

(1) G is isomorphic to $T(r, s, t)$ for $r \geq 4$;
(2) G is isomorphic to $T(1, s, 2)$ for $s \geq 9$, $s \neq 11$;
(3) G is isomorphic to $T(2, s, t)$ for s and t both even;
(4) G is 3-chromatic.

Moreover, the 5-list colorings can be given in linear time. Furthermore, none of these graphs are 3-choosable. Hence, $\chi_\ell(G) \in \{4, 5\}$ if any of the cases (1) to (4) hold for G.

We are currently unable to comment on the choosability of the excluded graphs, but we note that they consist only of nine nonisomorphic 5-chromatic graphs, as well as a subcollection of triangulations of the specific form $T(1, s, t)$ that are 4-chromatic. For any tuple (r, s, t), there is a simple formula describing each tuple (r', s', t') such that $T(r, s, t)$ is isomorphic to $T(r', s', t')$ (see [11]), and there are at most 6 such tuples for any (r, s, t). It is also not difficult to see that the loopless multigraphs $T(r, s, t)$ are all 5-choosable. So, in this sense, Theorem 1.10 covers the 5-choosability of "most" 6-regular toroidal triangulations. Furthermore, among those graphs covered in Theorem 1.10, the 5-chromatic ones are precisely those isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$. Thus, we have:

Corollary 1.11. *If G is isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$, $s \geq 9$, $s \neq 11$, then G is 5-chromatic-choosable, i.e. $\chi(G) = \chi_\ell(G) = 5$. Moreover, a 5-list coloring can be found in linear time.*

To the best of our knowledge, the method of proof that we employ is novel, in that we develop a framework that allows us to systematically compare the lists on vertices that are not too far apart, and that allows us to compute the list coloring in an efficient manner. By using the differential information between lists on nearby vertices, we reduce the *list configurations* that need to be considered. This kind of "list calculus" differs from other list coloring algorithms in the literature, which instead reduce the possible *graph configurations* by exploiting general structure results on the family of graphs under consideration (minimum girth, edge-width, etc.), while the specific lists on the graphs remain nebulous. Our method of proof could prove fruitful in other areas where a structure theorem—such as Theorem 1.1 in our case—allows one to shift attention towards the configuration of the lists themselves. We also emphasize that our linear-time algorithm for 5-list coloring these graphs is nearly best possible, since any fixed vertex needs to be "scanned" very few times.

1.3.3 Related work

Colorability vs. choosability

Note that it follows from Brooks's theorem for choosability that any 6-regular toroidal triangulation not isomorphic to K_7 is 6-choosable. Albertson and Hutchinson [2] showed that there is a unique simple graph in this family that is 6-chromatic, which has 11 vertices, and Thomassen [96] later classified all the 5-colorable toroidal graphs. But a precise characterization of all the 5-chromatic 6-regular toroidal triangulations was completed only recently by the combined work of several authors [32, 104], including the work discussed in this thesis in Chapter 4. Our results are the first in this line to attempt to characterize the list colorability of the 6-regular triangulations on the torus.

Choosability of grids

The problem of determining the choice number of 4-regular toroidal $m \times n$ grids, for $m, n \geq 3$, has been raised by Cai, Wang and Zhu [27]. These graphs are a special case of those satisfying $(d, m) = (4, 4)$. It is easy to show by induction that these grids are all 3-colorable, and the above authors conjecture that they are also 3-choosable. Recent work by Li, Shao, Petrov and Gordeev [64] has nearly determined the choice number of these grids as follows: if mn is even, then the choice number is 3, else it is either 3 or 4. Contrasting this with Theorem 1.10, we note that both nearly determine the choice number in the sense that the true value of the choice number is either equal to, or one less than, the computed value for each member of the family. However, their result does not a priori give an efficient algorithm for $\mathcal{L}$-coloring the toroidal grids since their proof uses the combinatorial nullstellensatz, whereas our result actually gives a linear time algorithm for $\mathcal{L}$-coloring the toroidal triangulations.

Recent algorithmic advances for list colorings

Dvořák and Kawarabayashi [37] have shown that there exists a polynomial time algorithm for 5-list coloring graphs embedded on a fixed surface. Postle and Thomas [80] have proved that for any surface Σ and every $k \in \{3, 4, 5\}$ there exists a linear time algorithm for determining whether or not an input graph G embedded in Σ and having girth at least $8 - k$ is k-choosable. In particular, when $\Sigma = S_1$ and $k = 5$, this implies that there is a linear time algorithm for determining whether or not any of the 6-regular triangulations under consideration are 5-choosable. This work was later extended by Postle in [79], wherein he showed that for each fixed surface Σ there exists a linear time algorithm to find a k-list coloring of a graph G with girth at least $8 - k$ for $k \in \{3, 4, 5\}$. Again, when $\Sigma = S_1$ and $k = 5$, this says that there is a linear time algorithm to find a 5-list coloring of a 6-regular triangulation on the torus.

Our results are stronger than those mentioned above for the class of 6-regular toroidal triangulations. Firstly, the high degree of the polynomial time algorithm in [37] makes it impractical to implement, though the authors suggest that it should likely be possible to reduce the bound enough to make the algorithm practical at least for planar graphs. Secondly, the linear time algorithm in [80] is contingent upon an enumeration of the 6-*list critical* graphs on the torus. Indeed, the authors show that there are only finitely many 6-list critical graphs on the torus, but a full list of these graphs is not explicitly known, and their bound on the maximum number of vertices any 6-list critical graph on the torus can have is far too large to be amenable to a straightforward enumerative check.[1] Also, their linear time algorithm does not specify an $\mathcal{L}$-coloring in the case when the graph is $\mathcal{L}$-choosable for a given list assignment $\mathcal{L}$. Thirdly, the linear time algorithm in [79] first requires a brute-force computation of the list colorings for any such list assignment on graphs of "small" order. However, the bound on the sizes of these small graphs is far too large to be computationally feasible, which makes the algorithm itself of mostly theoretical interest, as noted in a recent work by Dvořák and Postle [39].

This is in contrast with the results in Chapter 4, wherein the 5-choosable graphs identified in Theorem 1.10 can also be given 5-list colorings in linear time, unlike as in [80]. Furthermore, the non-3-choosability of the 3-chromatic graphs $T(r, s, t)$ is not covered by the results in [80] since these graphs have girth equal to 3, whereas their algorithm for 3-list coloring is applicable only for graphs having girth at least 5. Lastly, our proof of Theorem 1.10 supplies an implementable algorithm for 5-list coloring all the toroidal graphs under consideration without the need for running a brute-force check on any of them, in contrast with [79].

1.4 Colorability of the 6-regular triangulations of the torus

As mentioned in Section 1.3.3, even the classification of the colorability of the 6*-regular triangulations of the torus was completed only recently. In Chapter 5, we discuss the final piece of this classification, which involves the* 4*-colorability of these graphs, and state the classification result in full generality.*

[1] It is worth contrasting this with the corresponding colorability problem: while Thomassen [98] has shown that for every fixed surface there are only finitely many 6-*critical* graphs that embed on that surface, explicit lists of these 6-critical graphs are known only for the projective plane [1], the torus [96] and the Klein bottle [29, 57].

1.4.1 Background

Heawood's formula (Theorem 1.6) shows that every loopless toroidal graph is 7-colorable, and it is clear that any loopless 6-regular triangulation requires at least 3 colors for a proper coloring. Thus, $3 \leq \chi(T(r,s,t)) \leq 7$ for every loopless triangulation $T(r,s,t)$. Our focus will be on the 4-colorable triangulations $T(r,s,t)$, for the following reason. At one end, the classification of the 3-chromatic 6-regular triangulations follows a simple criterion: $T(r,s,t)$ is 3-chromatic if and only if $s \equiv 0 \equiv r-t \pmod 3$. At the other end, Dirac's map color theorem [35] implies that K_7 (which embeds isomorphically to $T(1,7,2)$ in the torus) is the only 7-chromatic 6-regular triangulation on the torus; moreover, Albertson and Hutchinson [2] showed that there is a unique simple 6-chromatic 6-regular triangulation on the torus, which has 11 vertices. So, it remains to classify the triangulations $T(r,s,t)$ that are 4- and 5-chromatic. Thus, a full description of the 4-colorable graphs $T(r,s,t)$ would complete the classification.

Collins and Hutchinson [32] gave a characterization of the 4-colorable triangulations $T(r,s,t)$ with $r,s \geq 3$ as follows:

Theorem 1.12 (Collins–Hutchinson [32, Theorem 1.2])**.** *Let $G = T(r,s,t)$. If $r,s \geq 3$, then G can be 4-colored, with a finite number of exceptions.*

The cases when $r \leq 2$ or $s \leq 2$ were partially resolved in the same work, and completed by Yeh and Zhu [104], ostensibly finishing the classification of the 4-colorability of the graphs $T(r,s,t)$.

1.4.2 Our work

In Chapter 5 of the thesis, we point out a gap in the proof of Theorem 1.12 that makes the statement incorrect, and we provide a patch to the statement and proof. To be precise, we first locate the error in the proof of Theorem 1.12, and provide explicit counterexamples to its statement. Then, we prove the following modification of the above theorem:

Theorem 1.13. *Let $G = T(r,s,t)$ be a simple 6-regular triangulation having normal circuits of lengths $a \geq b \geq c$. Suppose that $\left(\frac{n}{a}, \frac{n}{b}\right) \neq (1,1),(1,2)$, where $n = rs$ is the order of G. Then G can be 4-colored.*

Combined with the earlier results [2, 32, 35, 52] as well as the classification of the 4-colorable triangulations $T(1,s,t)$ by Yeh–Zhu [104], we complete the characterization of the colorability of all the 6-regular triangulations on the torus as follows.

Theorem 1.14. *Let $G = T(r,s,t)$ for $r \geq 1$, $s \geq 1$, $0 \leq t \leq s-1$ be a 6-regular triangulation on the torus. If $r = 1$, then $T(1,s,t)$ is isomorphic to $T(1,s,s-t-1)$, so in this case consider t only in the range $0 \leq t \leq \lfloor (s-1)/2 \rfloor$.*

1. *G contains loops if and only if either $s = 1$, or $r = 1$ and $s = 2$, or $r = 1$ and $t = 0$.*
2. *G is 7-chromatic if and only if G is isomorphic to K_7, and this happens only when $G = T(1, 7, 2)$.*
3. *G is 6-chromatic if and only if G is isomorphic either to K_6 (after deleting duplicated edges), or to the graph of Albertson and Hutchinson [2] on 11 vertices. The former happens only when $G \in \{T(1, 6, 2), T(2, 3, 0), T(2, 3, 1), T(3, 2, 0), T(3, 2, 1)\}$ and the latter only when $G \in \{T(1, 11, 2), T(1, 11, 3), T(1, 11, 4)\}$.*
4. *G is 5-chromatic if and only if G is one of the following graphs:*
   (a) *$T(1, 5, 1)$, $T(1, 5, 2)$ (these are isomorphic to K_5 after deleting duplicated edges);*
   (b) *$T(1, s, 2)$ for $s \geq 9$, $s \neq 11$, $s \not\equiv 0 \pmod 4$;*
   (c) *$T(1, s, t)$ for $s \in \{2t + 2, 2t + 3, 3t + 1, 3t + 2\}$, $s \geq 9$, $s \not\equiv 0 \pmod 4$;*
   (d) *$T(2, s, 0)$, $T(2, s, 1)$, $T(2, s, s - 3)$, $T(2, s, s - 2)$ for odd $s \geq 5$;*
   (e) *$T(3, s, s - 2)$, $T(3, s, s - 1)$ for $s \geq 3$, $s \not\equiv 0 \pmod 4$;*
   (f) *$T(r, 2, 0)$, $T(r, 2, 1)$ for odd $r \geq 5$;*
   (g) *$T(1, s, t)$ for $(s, t) \in \{(13, 3), (17, 3), (17, 4), (17, 6), (18, 3), (19, 7), (25, 3), (25, 6), (25, 7), (25, 9), (25, 10), (26, 7), (26, 10), (33, 6), (33, 14), (37, 10)\}$;*
   (h) *$T(2, s, t)$ for $(s, t) \in \{(9, 3), (9, 4), (13, 3), (13, 8)\}$;*
   (i) *$T(3, s, t)$ for $(s, t) \in \{(6, 1), (6, 2), (11, 2), (11, 6)\}$;*
   (j) *$T(5, s, t)$ for $(s, t) \in \{(5, 2), (5, 3)\}$.*
5. *G is 4-colorable in all other cases.*
6. *In particular, G is 3-chromatic if and only if $s \equiv 0 \equiv r - t \pmod 3$.*

Chapter 2

Classifying the 6-regular triangulations of the torus

Since we will be working closely with the 6-regular triangulations of the torus in the upcoming chapters, we start by giving a detailed proof of Theorem 1.1 due to Altshuler that classifies these embeddings.

2.1 Definitions

We shall define the graph theoretic terminology as explicitly as possible, to avoid ambiguities. We assume familiarity with the basic terminology of topological spaces.

2.1.1 Graphs

An *undirected graph* $G = (V, E)$ consists of a collection of *vertices* $v \in V$ and a collection of *edges* $e \in E$, where each edge is associated to a set of one or two vertices, which *lie on the edge*. The unqualified term *graph* shall always mean an undirected graph. We also say that an edge e is *incident* on a vertex v if the vertex lies on the edge. The *ends* of an edge are the vertices on which it is incident. Vertices v and w are *adjacent* to each other if they are both incident on a common edge. An edge that is incident on a single vertex v is also called a *loop* based at v. A graph has *multi-edges* if more than one edge has the same set of ends; such a graph is also called a *multigraph*. A graph without loops or multiple edges is said to be *simple*. For a simple graph, it is convenient to identify an edge with the set of its ends.

An *orientation* (or a *direction*) of an edge $e \in E$ is defined as follows. If e is not a loop, then it is an ordering of its set of ends; if v and w are the ends of e, then one of its ends is called the *head* and the other the *tail*, and the edge is said to be oriented from the tail to the head. If e is a loop, then it is a member of the two-element set $\{e\} \times \{\pm 1\}$. Thus, every edge has precisely two orientations.[1] An *orientation* $\overrightarrow{G}$ of a graph G is an assignment of an orientation to each edge; such a graph is also called a *directed* graph. For a simple directed graph, it is convenient to identify a directed edge e with the ordered pair (v, w) of its ends, where v is the tail and w the head.

For a graph G with an orientation $\overrightarrow{G}$, the *outdegree* and *indegree* of a vertex v are the number of edges with tail and head as v, respectively. For an undirected graph G, the *degree* of a vertex is the sum of its outdegree and indegree in any orientation $\overrightarrow{G}$. Thus, the degree of a vertex v is the number of edges incident on it, where each loop based at v accounts for two incidences at v.

A *walk* W of length $n \geq 0$ in a graph $G = (V, E)$ is a sequence

$$W = (v_0, e_{01}, v_1, e_{12}, \ldots, v_{n-1}, e_{n-1,n}, v_n),$$

where each edge $e_{i,i+1}$ is incident on v_i and v_{i+1}.[2] A walk *passes through* a vertex v (or edge e) if it occurs in the sequence defining the walk. The walk W is *closed* if $v_n = v_0$. A closed walk is also called a *circuit*. Two circuits of length n are considered to be identical if the cyclic sequences that they define are cyclic shifts of each other. The vertices v_i for $0 < i < n$ are called *internal vertices* of the walk W; when W is a circuit, the vertex $v_0 = v_n$ is also called an internal vertex. A *path* is a walk in which all edges and vertices are distinct. A *cycle* is a circuit in which all edges and vertices (except $v_0 = v_n$) are distinct. A *directed walk* is defined similarly, where instead of $e_{i,i+1}$ we have an oriented edge $d_{i,i+1}$ with tail as v_i and head as v_{i+1}, for each i; corresponding definitions can be made for directed circuits, paths, and cycles. For a simple graph, it is convenient to identify a walk with the subsequence of vertices that occur in it.

A graph G is *connected* if there is a walk passing through any finite set of vertices of G. The maximal connected subgraphs of G are its *connected components*.

2.1.2 Surfaces

A topological space X is called a *surface* if it is nonempty, Hausdorff, second-countable, and *locally homeomorphic* to $\mathbb{R}^2$; the latter condition means that for each point $x \in X$ there exists an open set $U_x \subset X$ containing x, as well as a map $\varphi_x \colon U_x \to \mathbb{R}^2$ that is a

[1]The purely combinatorial definition of orientation of an edge as an ordering of its ends will give a single orientation to a loop. However, it will be crucial for us that every edge, including any loop, have two orientations.

[2]One could omit the vertices in the sequence W, since they are completely determined by the sequence of edges that occur in W, but we choose to include them for our convenience.

homeomorphism. A surface X is *connected* if, whenever X is decomposed as the disjoint union of finitely many subspaces X_i, $i \in I$, we have $X = X_i$ for some i. The maximal connected subspaces of X are its *connected components*.

The following are examples of compact connected surfaces.

1. The *sphere* S_0, which can be viewed as the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ equipped with the subspace topology inherited from $\mathbb{R}^3$.
2. The *real projective plane* N_1, which can be viewed as a quotient of the sphere S_0 by the antipodal relation $(x, y, z) \sim (-x, -y, -z)$, equipped with the quotient topology.
3. The *torus* S_1, which can be viewed as the product $C \times C$ of two circles, equipped with the product topology.
4. Any *connected sum* $X \# Y$ of two compact connected surfaces X and Y, which is formed by deleting an open disc from each surface and gluing the boundary circles together.

The classification of compact connected surfaces goes as follows:

Theorem 2.1. *The following topological spaces are all compact connected surfaces that are pairwise non-homeomorphic: the sphere S_0, the connected sum of g tori for each $g \geq 1$, denoted S_g, and the connected sum of k real projective planes for each $k \geq 1$, denoted N_k. Furthermore, every compact connected surface is homeomorphic to one of the listed spaces.*

The *genus* of S_g, $g \geq 0$, is defined to be g, and of N_k, $k \geq 1$, is defined to be k. The *Euler characteristic* for these surfaces is defined as follows: $\chi(S_g) = 2 - 2g$, for $g \geq 0$, and $\chi(N_k) = 2 - k$, for $k \geq 1$.

2.1.3 Combinatorial embeddings

A *rotation system* for a connected graph $G = (V, E)$ is a tuple (D, σ, τ), where:

1. D is the set of directed edges of G (in particular, $|D| = 2|E|$);
2. σ is a permutation of D, which, for each $v \in V$, cyclically permutes the set D_v of directed edges with v as the tail;
3. τ is the involution on D that maps each directed edge to the one with the opposite orientation;
4. the group $\langle \sigma, \tau \rangle$ generated by σ and τ acts *transitively* on D, i.e., for every $d, d' \in D$ there exists $g \in \langle \sigma, \tau \rangle$ such that $gd = d'$.[3]

[3]This condition is imposed because the graph G is assumed to be connected.

A rotation system is also called a *(combinatorial) embedding*.

A *face* of a rotation system (D, σ, τ) for the graph G is an orbit of the permutation $\kappa := \sigma \circ \tau$ of D. The *degree* of a face is the size of the orbit that defines it. Note that if $d \in D$ has the vertex v as its head, then κd has v as its tail. So, for each $d \in D$, define a *facial circuit* of the face containing d to be any directed circuit in G defined by the orbit of d under κ. In particular, such a circuit looks like $(v_0, \kappa^i d, v_1, \kappa^{i+1} d, \ldots, v_{m-1}, \kappa^{i+m-1} d, v_0)$ for some $i \in \mathbb{N}$ and appropriate vertices v_i. A directed edge d (resp., vertex v) is *incident* (or *lies*) on a face f if d (resp., v) belongs to a facial circuit of f. Note that every $d \in D$ is incident on a unique face, and that d and τd could be incident on the same face. Also note that the degree of a face equals the length of any facial circuit of that face. If a facial circuit has no repeated vertices or edges, we call it a *facial cycle*.

A rotation system for a graph G, which need not be planar, can nevertheless be conveniently represented by a planar diagram. For a vertex v, the orbit of σ on the set D_v defines an ordering of the set of directed edges with tail as v. So, in a planar drawing of the rotation system, we shall draw the edges coming out of the vertex v in such a manner that their anti-clockwise ordering matches with that defined by the permutation σ. This gives a planar drawing (with crossing edges, in general) that fully represents the rotation system: for each directed edge d, say from v to w, the directed edge κd is found by picking the next edge in the anti-clockwise ordering of the edges around w in the planar drawing.

See Figure 2.2b for a planar drawing of a rotation system for the complete graph K_7. The rotation system represents the 6-regular triangulation $T(1, 7, 2)$ of the torus (see Figure 2.2a), which will be defined in Section 2.2.

Two combinatorial embeddings (D_i, σ_i, τ_i), $i \in \{1, 2\}$, are said to be *isomorphic* if there is a bijection $\varphi \colon D_1 \to D_2$ that maps orbits of σ_1 and κ_1 to corresponding orbits of σ_2 and κ_2, respectively.

2.1.4 Topological embeddings

Let us fix a compact connected surface X and a connected graph $G = (V, E)$. A *drawing* of G on X is a collection P of points in X, and a collection A of *arcs* on X (i.e., homeomorphic images of the closed interval $[0, 1]$ in X), which are in bijection with the sets V and E, respectively, and such that:

1. if the arc a is associated to the edge e, then the endpoints of the arc a (i.e., the images of 0 and 1 in X) are the points in P associated with the ends of e; furthermore, no other points in P belong to the arc a,
2. no two arcs intersect in their interiors (i.e., in the images of the open interval $(0, 1)$).

A drawing of G on X is also called a *(topological) embedding* of G into X. The subspace of X defined by the drawing of G is also called the image of G in X under the embedding, and it is convenient to identify G with its image in X, and then speak of G as a subset of X.

The connected components of $X - G$ are called the *faces* of the embedding. Thus, we shall sometimes use the notation $G = (V, E, F)$ to denote the embedding of a graph $G = (V, E)$ into X, with F as the set of faces of the embedding. An edge e (resp., vertex v) is *incident* (or *lies*) on a face f if its associated arc $a \in A$ (resp., point $p \in P$) on X is contained in the boundary of f. In particular, any edge lies on either one or two faces.

If each face of an embedding is homeomorphic to an open disc, then the embedding is called a 2-*cell embedding*. Given a 2-cell embedding (V, E, F), for any $f \in F$ there is a closed walk passing through all and only those vertices and edges that are incident on f. We refer to such a closed walk with minimum length as a *facial circuit*. If such a circuit is in fact a cycle, then we call it a *facial cycle*. Note that each edge occurs exactly twice in the collection of all facial circuits of an embedding.

Let $G = (V, E, F)$ be a 2-cell embedding into X. The *Euler–Poincaré formula* says that $\chi(X) = \mathsf{v} - \mathsf{e} + \mathsf{f}$. Two topological embeddings $G_i = (V_i, E_i, F_i)$, $i \in \{1, 2\}$, into a surface X are said to be *isomorphic* if there exists a homeomorphism $\varphi\colon X \to X$ that maps the drawing of G_1 into the drawing of G_2, such that the restriction induces an isomorphism of the underlying graphs.

2.1.5 Polyhedral decompositions of surfaces defined by graph embeddings

A 2-cell embedding of a graph G on a surface X defines a *polyhedron* that is homeomorphic to X, where a polyhedron is a connected topological space obtained from a disjoint union of polygons[4] by identifying pairs of edges (in particular, there must be an even number of edges in this disjoint union of polygons). Specifically, if $G = (V, E, F)$ is a 2-cell embedding of G into X, then to each $f \in F$ of degree m we associate an m-gon P_f whose vertices and boundary edges are labelled appropriately by a facial circuit of f. The polyhedron homeomorphic to X is defined by appropriately identifying edges with the same label; in particular, the end vertices should match whenever two edges are identified. This is called a *polyhedral decomposition* of the surface X, induced by the 2-cell embedding of G.

[4] We allow polygons to have 1 or 2 edges, too, though these cannot be represented as *convex* regions in the plane. So, one can choose to visualize any m-gon, $m \geq 1$, as a closed disk with m points appropriately picked out on the boundary circle.

Now, each polygon can be oriented either clockwise or anti-clockwise. A choice of orientation for a polygon induces orientations on each edge of the polygon. Suppose that the identification of a pair of edges (labelled e, say) is opposite to this induced orientation, in the sense that the edges are identified by matching their heads to their tails. Then, we relabel one of the edges to the symbol e^{-1}, so the two occurrences of this edge now have distinct labels. If there is a choice of orientation of each polygon so that, after relabelling, every edge occurs with distinct labels, then the resulting polyhedron is said to be *orientable*, else we call it *non-orientable*.[5]

2.1.6 Topological and combinatorial embeddings coincide on orientable surfaces

Each rotation system (D, σ, τ) naturally defines an embedding of $G = (V, E)$ into some oriented surface, as follows. To each (combinatorial) face, we associate a polygon that is oriented anti-clockwise and whose boundary cycle is labelled by the corresponding (combinatorial) facial circuit in the anti-clockwise direction. Since each directed edge occurs in a unique (combinatorial) face, this guarantees that the surface defined by such a polyhedral decomposition (by pasting together the edges labelled d and τd while matching their heads to their tails) is orientable. It is also clear that this procedure defines a 2-cell embedding of G into this surface.

Conversely, given a 2-cell embedding $G = (V, E, F)$ into an orientable surface X, we get a compatible rotation system for the graph in the following manner. Note that each (topological) facial circuit of the embedding defines a directed circuit in the graph. Let $D' \subset D$ be the set of all directed edges that occur in such directed circuits. We have a permutation $\theta \colon D' \to D'$ which sends each $d \in D'$ to the next edge in the directed circuit to which it belongs. The question remains, is $D' = D$? Since X is orientable, the 2-cell embedding (V, E, F) arises from a polyhedral decomposition in which every edge occurs with distinct labels. This guarantees that every directed edge occurs exactly once in the collection of all directed facial circuits. Hence, $D' = D$. Let $\sigma := \theta \circ \tau$. One can show that $\langle \sigma, \tau \rangle$ acts transitively on D, using the assumption that G is connected. Then, (D, σ, τ) is a rotation system for G that is compatible with the given 2-cell embedding of G into X.

Thus, every topological embedding is described by a combinatorial embedding, and every combinatorial embedding gives rise to a topological embedding. Furthermore, it is clear

[5] It is a fact that orientability is a topological invariant, so we say that the surface X is orientable if it has a polyhedral decomposition that is orientable. In particular, the surfaces S_0 and S_1 are orientable, as is any connected sum of tori, and the surface N_1 is non-orientable, as is any connected sum of projective planes.

from the description above that, in this correspondence, the combinatorial and topological definitions of faces, facial circuits, etc., all coincide. See [49] for further details.

2.2 6-regular triangulations of the torus

If every face of a 2-cell embedding of a graph G has degree 3, then the embedding is called a *triangulation*.[6] We now state the classification of 6-regular triangulations of the torus, due to Altshuler [11, 12]. For integers $r \geq 1$, $s \geq 1$ and $0 \leq t \leq s-1$, take $V = \{(i,j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ to be the vertex set of the graph $T(r,s,t)$ equipped with the following edges:

- For each $1 < i < r$, (i,j) is adjacent to $(i, j \pm 1)$, $(i \pm 1, j)$ and $(i \pm 1, j \mp 1)$.
- If $r > 1$, $(1,j)$ is adjacent to $(1, j \pm 1)$, $(2,j)$, $(2, j-1)$, $(r, j+t+1)$ and $(r, j+t)$.
- If $r > 1$, (r,j) is adjacent to $(r, j \pm 1)$, $(r-1, j+1)$, $(r-1, j)$, $(1, j-t)$ and $(1, j-t-1)$.
- If $r = 1$, $(1,j)$ is adjacent to $(1, j \pm 1)$, $(1, j \pm t)$ and $(1, j \pm (t+1))$.

Here, addition in the first coordinate is taken modulo r and in the second coordinate is taken modulo s. Figure 2.1 depicts a topological embedding of the graph $G = T(5,6,2)$ into the torus as a triangulation. (Here, the torus is viewed as a rectangular planar region with opposite sides appropriately identified.)

In general, we embed $T(r,s,t)$ into the torus as a triangulation in the following manner. Take the $(r+1) \times (s+1)$ plane grid graph, with vertices naturally labelled by $[r+1] \times [s+1]$. (Ignore the outer face, since we will be identifying the sides of this rectangle to obtain a torus.) Note that each (bounded) face in this drawing has a facial cycle of length 4. Triangulate each face by adding edges parallel to the line $x + y = 0$, joining a pair of non-adjacent vertices for every face. Now make the following identifications: for each $x \in [1, r+1]$, set $(x, s+1) \sim (x, 1)$, and then for each $y \in [1, s+1)$, set $(r+1, y) \sim (1, y-t)$, where the second coordinate is chosen modulo s to lie in $[1, s+1)$. Then, we obtain a 6-regular triangulation of the torus arising from an embedding of the graph $T(r,s,t)$.

By an abuse of notation, we shall also refer to this embedding into the torus as $T(r,s,t)$. Theorem 1.1 says that these are all the 6-regular triangulations on the torus up to iso-

[6] This terminology is slightly non-standard—usually, this term (or, more generally, the term *polygonization*) is reserved for 2-cell embeddings of *simple* graphs, whereas we also consider multigraphs with loops—but it should not cause any confusion for the purposes of this thesis. Similarly, the term *polyhedral decomposition* is sometimes reserved for those obtained from 2-cell embeddings of simple graphs, though we use it in a more general sense in this chapter.

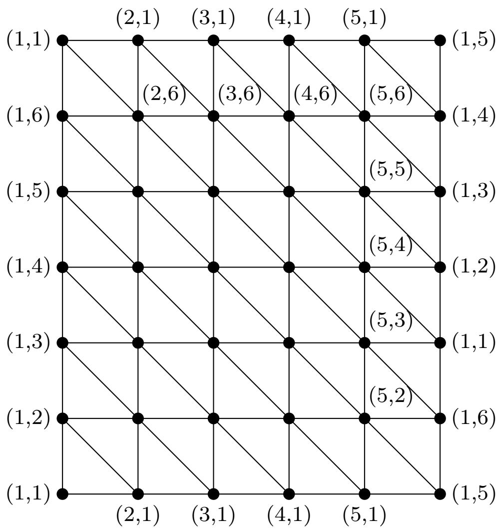


Figure 2.1: $G = T(5, 6, 2)$

morphism. A similar argument also appears in a paper by Negami [74]. We restate the theorem below.

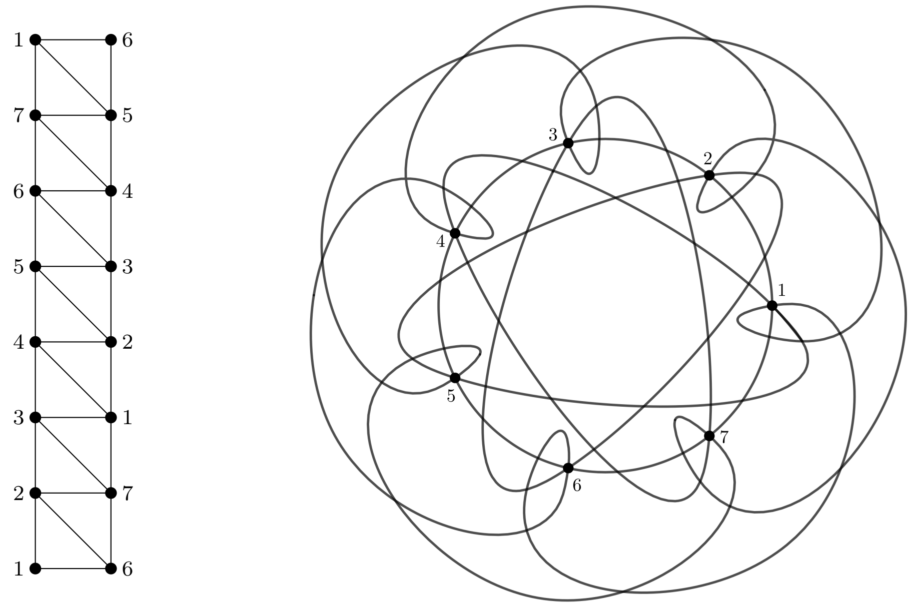
Theorem 1.1 (Altshuler [11, 12])**.** *Every 6-regular triangulation on the torus is isomorphic to $T(r, s, t)$ for some integers $r \geq 1$, $s \geq 1$, and $0 \leq t < s$.*

Outline of the proof. On the one hand, it is clear that every $T(r, s, t)$ is a 6-regular triangulation of the torus, by construction. To show, on the other hand, that any 6-regular triangulation of the torus is of this form, it is practically enough to show that we can cut open the torus along two cycles which correspond to a "vertical" and a "horizontal" cycle in $T(r, s, t)$.

So, pick an edge, and fix an end of the edge. Extend it to a walk that always goes "straight ahead" in the following sense: at each successive end v of the walk, pick the next edge such that there are exactly two edges on either side of the walk at the vertex v. Since there are finitely many edges, this walk eventually forms a circuit. If we can show that such a circuit is actually a cycle, then we will mostly be done.

This can be proved by contradiction: if there is such a circuit that is not a cycle, then there is a cycle that satisfies the same property at all but one of its vertices. Then, we show that we can find another cycle with the same property, but of smaller length. By verifying that no such cycles exist of small enough length, the proof will be (more-or-less) complete. □

The proof below will mostly use the description of an embedding as a rotation system in order to avoid loose appeals to geometric intuition. Though figures are supplied to assist



(a) A topological embedding **(b)** A combinatorial embedding

Figure 2.2: The triangulation $T(1, 7, 2)$ of the torus

one's intuition, the proofs do not use any additional geometric facts that are implied by those figures, since they need not hold in general.

Here is an illustration of some of the subtleties involved with the proof. Given a cycle satisfying a certain property, to show that there is a cycle of smaller length with the same property, one is tempted to visualize a neat, "locally planar" picture as in Figures 2.3 and 2.4. But this implicitly assumes that the vertices u, v, w, x, y in these pictures are all distinct, and moreover that there exists a drawing of these vertices along with their induced faces that has no crossing edges. However, only a diagram that represents a rotation system can accurately capture all of this information. Compare Figures 2.3 and 2.4 with Figure 2.2b, which represents a rotation system for an embedding of the complete graph K_7 into the torus as the 6-regular triangulation $T(1, 7, 2)$. (In Figure 2.2a, each label "$(1, j)$" is replaced by the label "j" for simplicity.)

Furthermore, the use of pictures such as Figures 2.3 and 2.4 tempt one to identify edges and faces by the vertices that occur in them, though this cannot be done when one is working with multigraphs, or graphs with loops. Indeed, both Altshuler and Negami happen to describe their proofs using the terminology of simple graphs, though they clearly have in mind the general picture of multigraphs with loops.

It should be clear now that arguments which involve such geometric intuitions (say, when we pick the "next" vertex on the "left" side of an edge) deserve to be made more precise. So, without further ado, let $G = (V, E, F)$ be a 6-regular triangulation of the torus, and let (D, σ, τ) be a rotation system corresponding to this embedding. For each $d \in D$, the orbit of d under σ has size 6, and the orbit of d under κ has size 3. We will use the notation $\mathsf{face}(d)$ to denote the face on which the directed edge $d \in D$ lies.

2.3 Proof of the classification of 6-regular triangulations of the torus

2.3.1 Preliminaries

We will require the following application of the Euler–Poincaré formula:

Lemma 2.2. *Let $G = (V, E, F)$ be a 2-cell embedding into the torus S_1, such that $\mathsf{degree}(f) \geq 3$ for every $f \in F$. Let $\bar{d} := 2\mathsf{e}/\mathsf{v}$ denote the average degree of G. Then, $\bar{d} \leq 6$, with equality holding if and only if the embedding is a triangulation of S_1.*

Proof. The Euler–Poincaré formula for the torus says that $\mathsf{v} - \mathsf{e} + \mathsf{f} = 0$. Since $\mathsf{degree}(f) \geq 3$ for all $f \in F$, we have $2\mathsf{e} \geq 3\mathsf{f}$ (use a double counting argument to find the cardinality of the set $\{(e, f) \in E \times F : e \text{ lies on } f\}$). It follows that $3\mathsf{v} - 3\mathsf{e} + 2\mathsf{e} \geq 0$, so $\bar{d} \leq 6$. Moreover, $\bar{d} = 6 \iff 2\mathsf{e} = 3\mathsf{f} \iff \mathsf{degree}(f) = 3$ for all $f \in F$, i.e., if and only if the embedding is a triangulation. □

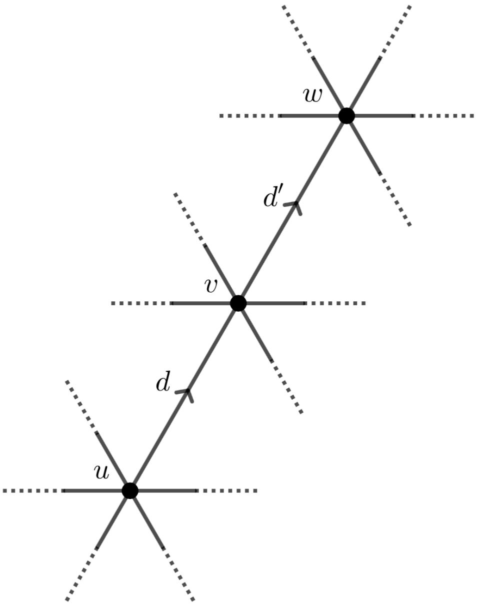
2.3.2 Normal walks

Let $W = (v_0, d_{01}, v_1, d_{12}, \ldots, v_{n-1}, d_{n-1,n}, v_n)$ be a directed walk in G of length $n \geq 1$. We say that W is *normal at the internal vertex v_i* if $d_{i,i+1} = \sigma^3\tau(d_{i-1,i})$. We also say that the vertex v_i is *normal* for the walk W when this happens. Informally, one imagines that a walk is normal at the vertex v_i if we happen to walk "straight ahead" through v_i. A walk itself is *normal* if it is normal at every internal vertex. Note that the walk W is normal if and only if its *reverse* walk $\overline{W} := (v_n, \tau d_{n-1,n}, v_{n-1}, \tau d_{n-2,n-1}, \ldots, v_1, \tau d_{01}, v_0)$ is normal.

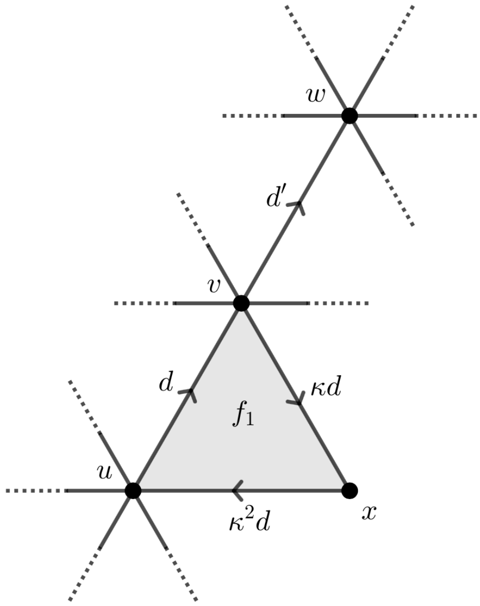
2.3.3 Normal walks can be parallelly translated

Let (u, d, v, d', w) be a walk for which v is normal. We do not assume that the vertices u and w are distinct, though we have chosen to display them as such in Figure 2.3 for the sake of convenience. In this figure, as well as all others in this section, assume that the edges are drawn such that σ permutes the edges incident on any vertex in an anti-clockwise fashion.

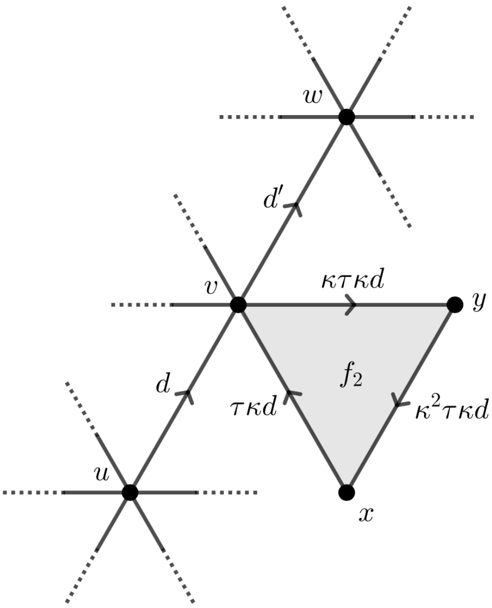
- Consider the face $f_1 = \mathsf{face}(\kappa d)$. Recall that $\kappa = \sigma\tau$. Since $\kappa^3 = \mathrm{id}$, f_1 has a facial cycle $(v, \kappa d, x, \kappa^2 d, u, d, v)$, for some vertex x. See Figure 2.3b.



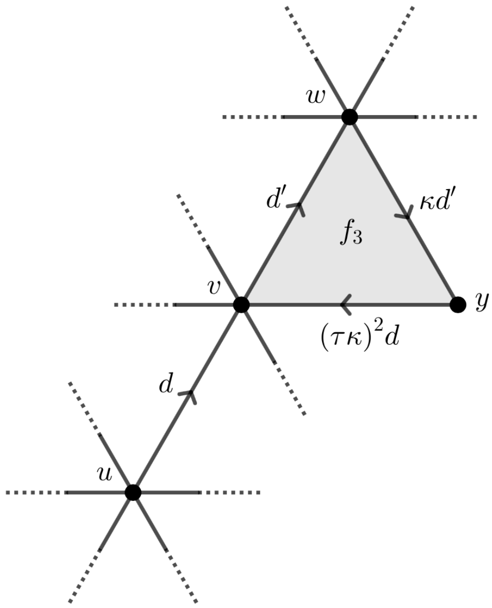
(a) The walk (u, d, v, d', w)



(b) The face $f_1 = \mathsf{face}(\kappa d)$



(c) The face $f_2 = \mathsf{face}(\tau\kappa d)$



(d) The face $f_3 = \mathsf{face}((\tau\kappa)^2 d)$

Figure 2.3: The walk (u, d, v, d', w), which is normal at v

- Next, consider the face $f_2 = \mathsf{face}(\tau\kappa d)$. Note that $\kappa\tau\kappa = \sigma^2\tau$. This face has a facial cycle $(x, \tau\kappa d, v, \kappa\tau\kappa d, y, \kappa^2\tau\kappa d, x)$, for some vertex y. See Figure 2.3c.
- Next, consider the face $f_3 = \mathsf{face}((\tau\kappa)^2 d)$. Note that $\kappa(\tau\kappa)^2 = \sigma^3\tau$. So, this face has a facial cycle $(y, (\tau\kappa)^2 d, v, d', w, \kappa d', y)$. See Figure 2.3d.

In particular, the edge $d'' = \tau\kappa^2\tau\kappa d$ is directed from x to y. We emphasize that no assumption is made that the vertices x and y are distinct from each other, or from the vertices u, v, and w.

Call the walk (x, d'', y) to be a *parallel translate* of the walk (u, d, v, d', w) in the *positive direction*. We have thus verified the following:

Lemma 2.3. *Let $W = (u, d, v, d', w)$ be a walk that is normal at v. Then, there exist vertices x, y, and a directed edge d'' from x to y, such that:*

- $\mathsf{face}(d)$ *has a facial cycle* $(u, d, v, \kappa d, x, \kappa^2 d, u)$;
- $\mathsf{face}(d')$ *has a facial cycle* $(v, d', w, \kappa d', y, \kappa^2 d', v)$;
- $d'' = \tau\kappa^2\tau\kappa d$;
- $\mathsf{face}(\tau d'')$ *has a facial cycle* $(y, \tau d'', x, \tau\kappa d, v, \kappa\tau\kappa d, y)$.

There is similarly a parallel translate in the *negative direction* by considering the face $\tilde{f}_1 = \mathsf{face}(\kappa\sigma d)$ instead of $f_1 = \mathsf{face}(\kappa d)$, and so on. We shall denote the parallel translate of W in the positive (resp., negative) direction by W^+ (resp., W^-). One can show that $\overline{W^-} = \overline{W}^+$, so it often suffices to establish results only for positive translates of normal cycles.

Note that $\kappa d'$ is incident on f_3, so we may define a parallel translate of a normal walk W of length $n \geq 2$ to be the walk obtained by applying the above procedure to successive subwalks of W of length equal to 2. Thus, if W has length $n \geq 2$, its parallel translate is a walk of length $n-1$ if W is not a circuit, and it is a circuit of length n if W is circuit.

Proposition 2.4. *Let $W = (v_0, d_{01}, v_1, d_{12}, \ldots, v_{n-1}, d_{n-1,n}, v_n)$ be a directed walk of length $n \geq 2$. If W is normal, then W^+ and W^- are also normal.*

Proof. It suffices to show that W^+ is normal. To do this, one has to check that

$$(\sigma^3\tau)(\tau\kappa^2\tau\kappa d_{i,i+1}) = \tau\kappa^2\tau\kappa d_{i+1,i+2}$$

for $0 \leq i \leq n-3$. And, when W is a circuit, one also has to check the above condition for $i = n-2$ and $i = n-1$. A straightforward calculation confirms these identities: we just use the fact that $d_{i+1,i+2} = \sigma^3\tau d_{i,i+1}$. □

Thus, given a normal walk W, one can speak of iterated parallel translates. Then, one can show that for a normal circuit W we have $(W^+)^- = W = (W^-)^+$.

2.3.4 Normal circuits are cycles

Say that a circuit W is *nearly normal* if it is not normal at exactly one vertex.

Proposition 2.5. *Every normal circuit is a cycle.*

Proof. We prove this proposition in two steps:

1. First, we show that if there is a circuit W that is not normal on at most one vertex, then either W is a normal cycle, or W has a subwalk $\widetilde{W}$ that is a nearly normal cycle.

   Thus, if there is a normal circuit that is not a cycle, then the latter case must hold.

2. Second, we show that if W is a nearly normal cycle, then a parallel translate of W or of $\overline{W}$ contains a nearly normal circuit of smaller length; however, we will also show there is no nearly normal cycle of length at most three.

   This contradiction implies that the latter case cannot hold, so every normal circuit is a cycle.

For the first step, let $W = (v_0, d_{01}, v_1, d_{12}, \ldots, v_{n-1}, d_{n-1,0}, v_0)$ be a circuit that is normal at v_i for all $1 \leq i \leq n-1$. Let $1 \leq k \leq n$ be least such that there exists $0 \leq j < k$ for which $v_j = v_k$ (here, $v_n \coloneqq v_0$). Take $\widetilde{W}$ to be the subwalk $(v_j, d_{j,j+1}, \ldots, v_{k-1}, d_{k-1,k}, v_k)$. If W is a normal cycle, then $\widetilde{W} = W$. If not, then $\widetilde{W}$ is a nearly normal cycle, since it is normal at every vertex except at $v_j = v_k$.

For the second step, suppose that $W = (v_0, d_{01}, v_1, d_{12}, \ldots, v_{n-1}, d_{n-1,0}, v_0)$ is a nearly normal cycle, where v_0 is not normal. Hence, we have $\sigma^i \tau d_{n-1,0} = d_{01}$ for some $i \in \{0, 1, 2, 4, 5\}$. Let $W^+ = (w_0, d'_{01}, w_1, d'_{12}, \ldots, w_{n-2}, d'_{n-2,n-1}, w_{n-1})$. By Lemma 2.3, we assume that

- $\mathsf{face}(d_{j,j+1})$ has a facial cycle $(v_j, d_{j,j+1}, v_{j+1}, \kappa d_{j,j+1}, w_j, \kappa^2 d_{j,j+1}, v_j)$ for each j;
- $d'_{j,j+1} = \tau \kappa^2 \tau \kappa d_{j,j+1}$ for each j.

Now, note that the cases when $i = 4$ and $i = 5$ can be handled by the cases when $i = 2$ and $i = 1$, respectively, by instead working with $\overline{W}$. So, we shall only consider the following three cases:

1. Case: $i = 0$. That is, $\tau d_{n-1,0} = d_{01}$.

So, such a cycle W must be a loop based at $v_0 = v$, i.e., $W = (v, d, v)$ for some directed loop d. But, the given condition says that $d = \tau d$, which contradicts that a loop has two orientations. Hence, $i = 0$ is impossible.

2. Case: $i = 2$. That is, $\sigma^2 \tau d_{n-1,0} = d_{01}$.

   Let $n \geq 2$. The given condition says that $\sigma\kappa d_{n-1,0} = d_{01}$, so $\tau\kappa d_{n-1,0} = \kappa^2 d_{0,1}$. Hence, $\mathsf{face}(d_{01}) = \mathsf{face}(\kappa^2 d_{01}) = \mathsf{face}(\tau\kappa d_{n-1,0})$. Now, $\mathsf{face}(d_{01})$ has a facial cycle

   $$(v_0, d_{01}, v_1, \kappa d_{01}, w_0, \kappa^2 d_{01}, v_0) = (v_0, d_{01}, v_1, \kappa d_{01}, w_{n-1}, \tau\kappa d_{n-1,0}, v_0).$$

   Hence, $w_0 = w_{n-1}$, and W^+ is a closed walk of length $n - 1$. See Figure 2.4a. Again, we do not assume that v_{n-1} and v_1 are distinct vertices, nor that $w_0 = w_{n-1}$ is distinct from the vertices v_i, though we display them as such for the sake of convenience.

   It only remains to show that W^+ is not normal at $w_0 = w_{n-1}$, i.e., that $d'_{01} \neq \sigma^3 \tau d'_{n-2,n-1}$.

   Lemma 2.6. $d'_{01} = \sigma^2 \tau d'_{n-2,n-1}$.

   Proof. Since $d'_{j,j+1} = \tau\kappa^2\tau\kappa d_{j,j+1}$ for all j, and $d_{01} = \sigma^2\tau d_{n-1,0}$, and $d_{n-1,0} = \sigma^3 \tau d_{n-2,n-1}$, we have that $d'_{01} = \sigma^2\tau d'_{n-2,n-1} \iff (\tau\kappa^2\tau\kappa)(\sigma^2\tau)(\sigma^3\tau) d_{n-2,n-1} = (\sigma^2\tau)(\tau\kappa^2\tau\kappa) d_{n-2,n-1}$, which is routinely verified. □

   Hence, W^+ is a nearly normal circuit of length $n - 1$, and it is of the same kind as W.

   Thus, we inductively arrive at a nearly normal cycle $\widetilde{W} = (v, d, v)$ for which $\sigma^2\tau d = d$. Since we must also have $\kappa^3 d = d$, we get $\sigma(\kappa d) = \sigma(\tau\kappa(\kappa d))$, so $d_1 = \tau\kappa d_1$ for $d_1 := \kappa d$. Hence, $\tau d_1 = \sigma(\tau d_1)$, which implies that $d_2 = \sigma d_2$, where $d_2 := \tau d_1$, and this is a contradiction.

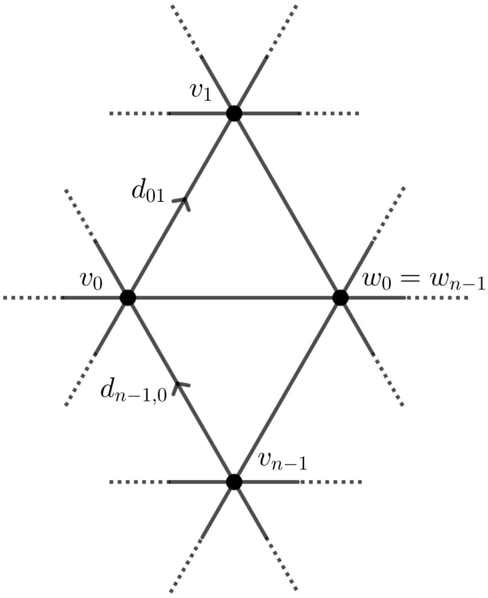
3. Case: $i = 1$. That is, $\sigma\tau d_{n-1,0} = d_{01}$.

   Let $n \geq 4$. The given condition says that $\kappa d_{n-1,0} = d_{01}$ and $\kappa^2 d_{01} = d_{n-1,0}$. Hence, $\mathsf{face}(d_{n-1,0}) = \mathsf{face}(\kappa d_{n-1,0}) = \mathsf{face}(d_{01})$. Now,

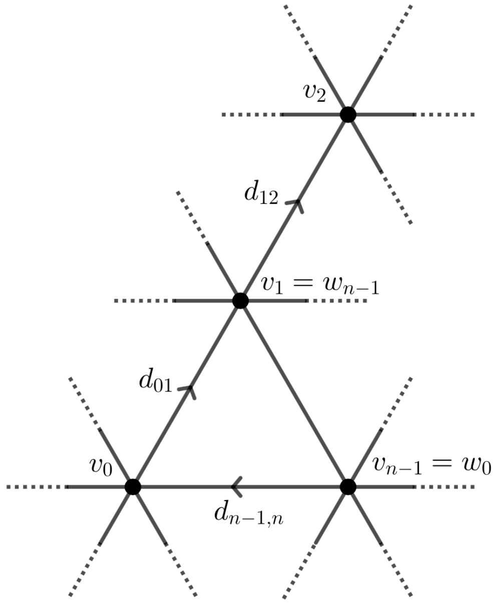
   - $\mathsf{face}(d_{n-1,0})$ has a facial cycle

     $$\begin{aligned}&(v_{n-1}, d_{n-1,0}, v_0, \kappa d_{n-1,0}, w_{n-1}, \kappa^2 d_{n-1,0}, v_{n-1})\\ &= (v_{n-1}, d_{n-1,0}, v_0, d_{01}, v_1, \kappa d_{01}, v_{n-1}),\end{aligned}$$

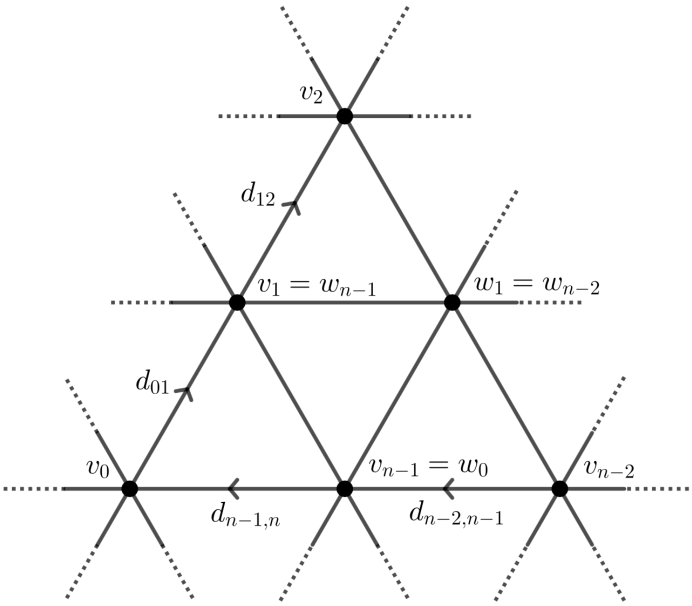
     so $w_{n-1} = v_1$.



(a) The case $\sigma^2\tau d_{n-1,0} = d_{01}$



(b) The case $\sigma\tau d_{n-1,0} = d_{01}$



(c) The case $\sigma\tau d_{n-1,0} = d_{01}$ (extended)

Figure 2.4: Nearly normal circuits

- $\mathsf{face}(d_{01})$ has a facial cycle

$$(v_0, d_{01}, v_1, \kappa d_{01}, w_0, \kappa^2 d_{01}, v_0) = (v_0, d_{01}, v_1, \kappa d_{01}, v_{n-1}, d_{n-1,0}, v_0),$$

  so $w_0 = v_{n-1}$.

See Figure 2.4b.

Next:

- Consider $\mathsf{face}(\tau\kappa d_{01})$. This has a facial cycle

$$\begin{aligned} &(w_0, \tau\kappa d_{01}, v_1, \kappa\tau\kappa d_{01}, w_1, \kappa^2\tau\kappa d_{01}, w_0) \\ =& (v_{n-1}, \tau\kappa d_{01}, v_1, \kappa\tau\kappa d_{01}, w_1, \kappa^2\tau\kappa d_{01}, v_{n-1}). \end{aligned}$$

- Consider $\mathsf{face}(\tau\kappa d_{n-2,n-1})$. This has a facial cycle

$$\begin{aligned} &(w_{n-2}, \tau\kappa d_{n-2,n-1}, v_{n-1}, \kappa\tau\kappa d_{n-2,n-1}, w_{n-1}, \kappa^2\tau\kappa d_{n-2,n-1}, w_{n-2}) \\ =& (w_{n-2}, \tau\kappa d_{n-2,n-1}, v_{n-1}, \kappa\tau\kappa d_{n-2,n-1}, v_1, \kappa^2\tau\kappa d_{n-2,n-1}, w_{n-2}). \end{aligned}$$

Notice that $\tau\kappa d_{01}$ and $\kappa\tau\kappa d_{n-2,n-1}$ are both directed edges from v_{n-1} to v_1. We claim that they are identical.

Lemma 2.7. $\kappa\tau\kappa d_{n-2,n-1} = \tau\kappa d_{01}$.

Proof. Since $d_{01} = \kappa d_{n-1,0}$, and $d_{n-1,0} = \sigma^3\tau d_{n-2,n-1}$, this boils down to the identity $(\kappa\tau\kappa)d_{n-2,n-1} = (\tau\kappa^2\sigma^3\tau)d_{n-2,n-1}$, which is routinely verified. □

Hence, $\mathsf{face}(\tau\kappa d_{01}) = \mathsf{face}(\tau\kappa d_{n-2,n-1})$. Comparing the above descriptions of their facial cycles, we get $w_1 = w_{n-2}$. See Figure 2.4c. Note that no assumption is made that v_2 and v_{n-2} are distinct, nor that the vertex $w_1 = w_{n-2}$ is distinct from the v_i, though we have chosen to display them as such in Figure 2.4c for the sake of convenience.

Thus, the subwalk $\widetilde{W}$ of W^+ from w_1 to w_{n-2} is a nearly normal circuit of length $n-3$. So, we inductively arrive at a nearly normal cycle of length $n \in \{1, 2, 3\}$, and we may assume that it is of the same kind as W.

(a) Case: $n = 1$.

  Let $\widetilde{W} = (v, d, v)$, and suppose that $\sigma\tau d = d$. Then, $d = \kappa d$, so $\mathsf{face}(d)$ has degree 1, which is a contradiction.

(b) Case: $n = 2$.

Let $\widetilde{W} = (v, d, w, \sigma^3\tau d, v)$ where $\sigma\tau\sigma^3\tau d = d$. Since we must also have $\kappa^3 d = d$, we get $\kappa\sigma^2\kappa d = \kappa^3 d$, i.e., $\sigma(\sigma\tau)(\tau\kappa d) = \sigma(\tau\kappa d)$, i.e., $\kappa d_1 = d_1$, for $d_1 := \tau\kappa d$. This gives us a contradiction as seen in the previous case.

(c) Case: $n = 3$.

Let $\widetilde{W} = (u, d, v, \sigma^3\tau d, w, (\sigma^3\tau)^2 d, u)$ where $\sigma\tau(\sigma^3\tau)^2 d = d$. Since we must also have $\kappa^3 d = d$, we get $\kappa(\sigma^3\tau)^2 d = \kappa^3 d$, i.e., $\sigma^3\tau\sigma^3\tau d = \sigma\tau\sigma\tau d$, i.e., $\sigma^2\tau\sigma^2\tau(\tau\kappa d) = \tau\kappa d$. Let $d_1 = \tau\kappa d$. Again, since $\kappa^3 d_1 = d_1$, we get $(\sigma^2\tau)^2 d_1 = \kappa^3 d_1$, i.e., $\sigma^2\tau\sigma(\sigma\tau d_1) = \sigma\tau\sigma\tau(\sigma\tau d_1)$, i.e., $\sigma\tau\sigma d_2 = \tau\sigma\tau d_2$, where $d_2 := \sigma\tau d_1$. Thus, we get $\tau\sigma\tau\sigma\tau(\tau d_2) = \sigma(\tau d_2)$. Let $\tau d_2 := d_3$. Then, we get $(\sigma\tau)^3 d_3 = \sigma^2 d_3$, i.e., $\kappa^3 d_3 = \sigma^2 d_3$, i.e., $d_3 = \sigma^2 d_3$, which is a contradiction.

□

2.3.5 Further properties of parallel translates

At this point, we have verified the steps mentioned in the outline of the proof in Section 2.2. We would like to finish off the proof by cutting open the torus along two normal cycles passing through a given vertex, and observing that the resulting picture has exactly the same description as that of $T(r, s, t)$ in Section 2.2, for some values of r, s, and t. To be able to do so, we need to verify that normal cycles behave as expected. More precisely:

1. We need to verify that what we call a parallel translate of W truly is "parallel": if W and W^+ (or W^-) are distinct, then they should not intersect. This is the content of Proposition 2.9.

2. Furthermore, successive parallel translates should also have the same property. That is, the set of all iterated parallel translates of any normal cycle W should consist of a collection of disjoint cycles. This is the content of Proposition 2.10.

This section can be viewed as an elaboration of remarks made in Negami [74].

Lemma 2.8. *If W is a normal cycle, and W' is a parallel translate of W, then $W' \neq \overline{W}$.*

Proof. Let W be a normal cycle of length $n \geq 1$. It suffices to prove this for $W' = W^+$. Let d be any directed edge on W. Suppose, for the sake of contradiction, that $W^+ = \overline{W}$. Then, $\tau\kappa^2\tau\kappa d$ lies on $\overline{W}$, so there exists $j \in \{0, 1, \ldots, n-1\}$ such that $\tau\kappa^2\tau\kappa d = \tau(\sigma^3\tau)^j d$.

Notice that we can always choose d so that $j \in \{0, 1\}$, which can be seen as follows. Let $W = (v_0, d_0, v_1, d_1, v_2, \ldots, v_{n-1}, d_{n-1}, v_0)$, and suppose that $\tau\kappa^2\tau\kappa d_0 = \tau(\sigma^3\tau)^j d_0 = \tau d_j$. Then, $W^+ = (v_{j+1}, \tau d_j, v_j, \tau d_{j-1}, v_{j-1}, \ldots, v_{j+2}, \tau d_{j+1}, v_{j+1})$. Comparing the sequences defining W and W^+, we see that $\tau\kappa^2\tau\kappa d_i = \tau d_{j-i}$ for all $0 \leq i \leq n-1$ (the subscript is taken modulo n on the RHS). Thus, if j is even, then there exists i such that $\tau\kappa^2\tau\kappa d_i =$

τd_i, and if j is odd, then there exists i such that $\tau\kappa^2\tau\kappa d_i = \tau d_{i+1} = \tau(\sigma^3\tau)d_i$. So, there is no loss of generality in assuming that d is chosen such that $j \in \{0, 1\}$. Now,

- if $j = 0$, then $\tau\kappa^2\tau\kappa d = \tau d$, so we have $\tau(\kappa d) = \kappa d$;
- if $j = 1$, then $\tau\kappa^2\tau\kappa d = \tau(\sigma^3\tau)d$, so we have $\tau(\sigma\kappa d) = \sigma\kappa d$.

In either case, we have a contradiction, since every edge has two orientations. □

Proposition 2.9. *Let W be a normal cycle, and let W' be a parallel translate of W. Then, either $W = W'$, or W and W' are disjoint cycles.*

Proof. Let $W = (v_0, d, v_1, \sigma^3\tau d, v_2, \dots, v_{n-1}, (\sigma^3\tau)^{n-1}d, v_0)$. It suffices to prove this for $W' = W^+$. Suppose that W^+ intersects W. Without loss of generality, suppose that $\mathsf{face}(d)$ has a facial cycle $(v_0, d, v_1, \kappa d, v_j, \kappa^2 d, v_0)$ for some $0 \le j \le n-1$. One may find Figure 2.3 to be a useful aid in following along with this proof.

Consider the subpath $(v_{j-1}, (\sigma^3\tau)^{j-1}d, v_j, (\sigma^3\tau)^j d, v_{j+1})$ of W. So, on the one hand, out of the six directed edges having tail as v_j, one of them has to be $(\sigma^3\tau)^j d$. On the other hand, we can view v_j as a vertex on W^+, and in particular as a vertex incident on $\mathsf{face}(d)$ as described above. So, we can describe the six directed edges with tail at v_j as $\sigma^i\tau\kappa^2\tau\kappa d$, $i \in \{0, 1, 2, 3, 4, 5\}$, and one of them must be equal to $(\sigma^3\tau)^j d$.

Notice that if $i = 0$, then $(\sigma^3\tau)^j d = \tau\kappa^2\tau\kappa d$. But then W and W^+ share a directed edge. Since there is a unique normal cycle obtained by extending this edge, we get $W = W^+$. Therefore, it suffices to show that in every other case we obtain a contradiction.

- Case: $i = 1$. That is, $(\sigma^3\tau)^j d = \sigma\tau\kappa^2\tau\kappa d = \tau\kappa d$. Note that $\tau\kappa d$ is directed from v_j to v_1, and $(\sigma^3\tau)^j d$ is directed from v_j to v_{j+1}. Hence, $v_1 = v_{j+1}$, so $j = 0$. But, this implies that $d = \tau\kappa d$, i.e., $\tau d = \sigma(\tau d)$, which is a contradiction.

  We may analyze the cases $i = 2$, 4, and 5 in a similar fashion, so we omit the details. The case $i = 3$ merits further remarks:

- Case: $i = 3$. Suppose that $(\sigma^3\tau)^j d = \sigma^3\tau\kappa^2\tau\kappa d$, so that $\tau(\sigma^3\tau)^{j-1}d = \tau\kappa^2\tau\kappa d$. Then, $W^+ = \overline{W}$, which contradicts Lemma 2.8.

Hence, we must have $i = 0$, so $W^+ = W$. □

For a normal cycle W, consider the set $\mathcal{C}(W)$ of normal cycles generated by W by taking parallel translates, $\{\dots, (W^-)^-, W^-, W, W^+, (W^+)^+, \dots\}$. Since there are finitely many edges in G, this is a finite set of cycles, say of cardinality r. Let $\mathcal{C}^+(W) = (W_1, W_2, \dots, W_r)$, where $W_1 := W$ and $W_{k+1} := (W_k)^+$ for $1 \le k \le r$, and similarly define $\mathcal{C}^-(W)$. (Here, we assume $W_{r+1} := W_1$.)

Proposition 2.10. *If W is a normal cycle, then $\mathcal{C}(W)$ consists of pairwise disjoint cycles.*

Proof. Suppose, for the sake of contradiction, that there exists a normal cycle that does not satisfy the statement of the above proposition. For such a normal cycle W, let $\mathcal{C}^-(W) = (W_1, \ldots, W_r)$, $r \geq 2$. Additionally, choose W for which the parameter $m(W)$ is minimal, where $m(W) := \min\{j \in \{2, 3, \ldots, r\} : W_1 \text{ and } W_j \text{ intersect}\}$. We claim that $m = 2$ for any such minimal cycle W.

Suppose, for the sake of contradiction, that $m > 2$ for such a minimal cycle W. Let $W = (v_0, d, v_1, \sigma^3\tau d, v_2, \ldots, v_{n-1}, (\sigma^3\tau)^{n-1}d, v_0)$. Without loss of generality assume that W_m intersects W at v_0. Again, Figure 2.3 may serve as a useful aid in this proof.

Note that $W_{m-1} = (W_m)^+$. By the definition of m, the cycles W and W_{m-1} are disjoint. Let (w', d', v_0, d'', w'') be a subwalk of W_m, so that $d'' = \sigma^3\tau d'$. Since $m \leq r$, we have $d'' \neq d$. So, $d'' = \sigma^i d$ for some $i \in \{1, 2, 3, 4, 5\}$. But, each of these choices results in a contradiction.

For instance, if $i = 1$, then one observes that $\tau\kappa^2\tau\kappa d''$ is an edge of $(W_m)^+$ whose tail coincides with the head of d, which is the vertex v_1. Hence, W and W_{m-1} intersect at v_1, which is a contradiction. One arrives at similar contradictions in the cases $i = 2$, 4, and 5, so we omit the details.

As in the previous proof, the case $i = 3$, merits a closer look. Suppose $d'' = \sigma^3 d = (\sigma^3\tau)(\tau d)$, so $W_1 = \overline{W_m}$. Then $W_2 = (W_1)^- = \left(\overline{W_m}\right)^- = \overline{(W_m)^+} = \overline{W_{m-1}}$, so W_2 and W_{m-1} intersect. If $m \geq 4$, then this contradicts that our choice of W was minimal for the parameter m, since $m(W_1) = m(W) - 2$. If $m = 3$, then we get $W_2 = \overline{W_2}$. This means that there are directed edges d and d' that lie on W_2 such that $d' = \tau d$. If $d \neq d'$, W_2 must intersect itself, so it is not a cycle, but this contradicts Proposition 2.5. If $d = d'$, then this contradicts that every edge has two orientations.

Hence, $m = 2$. So, W and $W^- = W_2$ intersect, but are not identical by assumption. This contradicts Proposition 2.9, so we are done. □

2.3.6 Proof of the main theorem

Let $G = (V, E, F)$ be a 6-regular triangulation of the torus, with (D, σ, τ) as the associated rotation system. Fix a vertex in V, and label it v_{11}. Let W be a normal cycle through v_{11}, say $W = (v_{11}, d_1, v_{12}, \sigma^3\tau d_1, v_{13}, \ldots, v_{1s}, (\sigma^3\tau)^{s-1}d_1, v_{11})$. Consider $\mathcal{C}^+(W) = (W_1, \ldots, W_r)$. Label W_j as $W_j = (v_{j1}, d_j, v_{j2}, \sigma^3\tau d_j, v_{j3}, \ldots, v_{js}, (\sigma^3\tau)^{s-1}d_j, v_{j1})$, where $d_j = \tau\kappa^2\tau\kappa d_{j-1}$ for each $2 \leq j \leq s$. Since $(W_r)^+ = W_1$, we have $(\tau\kappa^2\tau\kappa)(\sigma^3\tau)^t d_r = d_1$ for some $0 \leq t \leq s - 1$. Note that Proposition 2.10 assures us that these rs vertex labels are placed on rs distinct vertices.

Using Lemma 2.3, we see that for every vertex labelled v_{ij}, $i \in [r]$, $j \in [s]$, we have accounted for all six edges incident at that vertex. Since $\langle \sigma, \tau \rangle$ acts transitively on D, we see that we have also accounted for all the vertices of the graph G. We now claim that the given embedding is isomorphic to $T(r, s, t)$. To show this, it suffices to observe that we obtain precisely the description of the embedding $T(r, s, t)$ given in Section 2.2, by first cutting open the torus along W, and then along the walk $P = (v_{11}, \sigma^{-1}d_1, v_{21}, \sigma^{-1}d_2, v_{31}, \ldots, v_{r1}, \sigma^{-1}d_r, v_{1,1-t})$. (Note that the expression $1 - t$ is taken modulo s, in order to lie in $[s]$.) Again, using Lemma 2.3 we see that the faces of this embedding match with the faces of $T(r, s, t)$ as described.

The only hiccup is that one first needs to be assured that the normal cycle W is non-contractible, otherwise cutting the torus along W and then P will not produce a plane rectangular region. This can be justified in the following way.[7] Suppose, for the sake of contradiction, that W is contractible. Consider that region of $S_1 \setminus W$ which is homeomorphic to an open disc, and call it B. Delete all the edges and vertices that lie in B,[8] add one new vertex in B, and add edges joining this new vertex to all the vertices on W. The new graph thus obtained is also a triangulation of the torus, so its average degree must be equal to 6 by Lemma 2.2. However, the average degree $\bar{d}$ of the new graph turns out to be less than 6, because if there are k vertices that do not lie on W or in B, then

$$\bar{d} = \frac{6k + 5s + s}{k + s + 1} = 6 - \frac{1}{k + s + 1} < 6.$$

This is a contradiction, so any normal cycle is non-contractible. $\square$

This completes the classification of 6-regular triangulations of the torus.

2.4 Further remarks

- It is clear from the description of $T(r, s, t)$ that the graph is vertex-transitive. In fact, the translation that sends each vertex (i, j) to $(i + i_0, j + j_0)$, for a fixed (i_0, j_0), extends to an isomorphism of the embedding $T(r, s, t)$ with itself.
- The proof of the main theorem shows that any 6-regular triangulation of the torus has up to six representations of the form $T(r, s, t)$.

  For the embedding $T(r, s, t)$, the normal cycles along each of the three directions—vertical, horizontal, and diagonal—have lengths s, $\frac{n}{\gcd(s,t)}$, and $\frac{n}{\gcd(s,r+t)}$, respectively, where $n = rs$ is the order of $T(r, s, t)$. Thus, a necessary condition for two 6-

[7] This argument follows Negami [74].
[8] Note that any loop that intersects B must lie completely inside B.

regular triangulations of the torus on n vertices to be isomorphic is that the triple (ℓ_V, ℓ_H, ℓ_D) of lengths of the normal cycles in the vertical, horizontal, and diagonal directions through any vertex be identical upto permutation. For example, the embeddings $T(3,3,0)$ and $T(3,3,1)$ are not isomorphic, since the former has the triple of lengths $(3,3,3)$, and the latter has the triple $(3,9,9)$.

- Consider the embedding $T(r,s,t)$. By picking a different normal circuit to be represented as the vertical cycle, one can see that there exist integers $0 \leq t_1 < \frac{n}{\gcd(s,t)}$ and $0 \leq t_2 < \frac{n}{\gcd(s,r+t)}$ such that $T(r,s,t)$ is isomorphic to $T\big(\gcd(s,t), \frac{n}{\gcd(s,t)}, t_1\big)$ as well as to $T\big(\gcd(s,r+t), \frac{n}{\gcd(s,r+t)}, t_2\big)$. By swapping the horizontal and diagonal normal circuits, one can see that $T(r,s,t)$ is isomorphic to $T(r,s,t')$ for $0 \leq t' < s$ such that $t' \equiv -r-t \pmod{s}$.

  Explicit formulas for t_1 and t_2 are provided in [11,12,74]. These formulas show that having the same triple of lengths of normal cycles is not sufficient for two embeddings to be isomorphic. For instance, Altshuler [12, page 208] remarks that $T(1,20,3)$ and $T(1,20,8)$ are not isomorphic despite that they have triples of lengths as $(20,20,5)$ and $(20,5,20)$, respectively. Similarly, Negami [74, Table 1] lists the embeddings $T(1,13,2)$ and $T(1,13,3)$ in distinct isomorphism classes, and it is easy to check that every normal cycle has length 13 in both embeddings (also see [12, Consequence 3.1]).

- The following instance of the above phenomenon will be important for us in Chapter 5. Suppose that the shift t is equal to 0. Then, it is easy to see that $T(r,s,0)$ and $T(s,r,0)$ are isomorphic embeddings. However, if t is not equal to 0, it can (and often does) happen that $T(r,s,t)$ is not isomorphic to $T(s,r,t')$ for any $0 \leq t' < s$.

- Negami [74, Theorem 4.3] shows that every simple graph $T(r,s,t)$ is uniquely embeddable in the torus, in the sense that two such graphs are isomorphic if and only if their 2-cell embeddings in the torus (which are necessarily triangulations by the Euler–Poincaré formula) are isomorphic. Negami remarks that the graphs $T(r,s,t)$ that have loops or multi-edges clearly admit 2-cell embeddings in the torus that are not triangulations, so the uniqueness of their embeddings is out of the question.

  However, one can still ask whether the notation $T(r,s,t)$ can be used to unambiguously refer to the underlying multigraph of the corresponding triangulation of the torus; that is, is it possible that the multigraph $T(r,s,t)$ is isomorphic to $T(r',s',t')$ even though their embeddings in the torus *as triangulations* are non-isomorphic? The answer turns out to be yes, but only in the following case. When r is even, the embeddings of $T(r,2,0)$ and $T(r,2,1)$ in the torus as triangulations are not isomorphic. One can see this by noting that the triple of lengths for the former is $(2,r,r)$, and for the latter is $(2,2r,2r)$. However, their underlying multigraphs turn out to be isomorphic. Any other graph $T(r, s, t)$ is isomorphic to $T(r', s', t')$ if and only if their embeddings in the torus as triangulations are isomorphic.

Chapter 3

Gap between the chromatic and list chromatic numbers

3.1 Preliminaries

We begin with the following result on the embeddability of the complete graph K_r into an orientable surface:

Theorem 3.1 (Ringel–Youngs [87]). *For every $r \geq 1$, K_r is embeddable in $S_{\gamma(r)}$ for $\gamma(r) = \lceil (r-3)(r-4)/12 \rceil$, and this is best possible.*

The above result, called the Ringel–Youngs theorem, also shows that $K_{H(g)}$ is embeddable in S_g for all $g \geq 1$, so it implies that Heawood's upper bound is tight for all $g \geq 1$.

Definition 3.2. Let $G = (V, E)$ be a simple graph, and $k \geq 1$. The *k-core* of G is the unique maximal subgraph of G having minimum degree at least k.

The next result is folklore; it was observed for $k = 2$ by Erdős, Rubin and Taylor [40], but we include the proof for general k for the sake of completeness.

Proposition 3.3. *Let $G = (V, E)$ be a simple graph, and $k \geq 1$. Then, G is k-choosable if and only if its k-core is k-choosable.*

Proof. If G is k-choosable, then so is every subgraph. In particular, its k-core is k-choosable. This proves the forward direction. For the reverse direction, suppose for the sake of contradiction that the k-core of G is k-choosable, but G is not k-choosable. Let $H \leq G$ be a subgraph of G with the least order such that H is not k-choosable. Note that H must be nonempty.

We show that the minimum degree of H must be at least k. Suppose for the sake of contradiction that there exists a vertex $v \in V(H)$ such that $\mathsf{degree}(v) < k$. Let $\mathcal{L}$ be a

k-list assignment on G for which H is not $\mathcal{L}$-choosable. The subgraph $H-v$ is k-choosable by the minimality of H with respect to the number of vertices, so properly color $H-v$ using the list assignment $\mathcal{L}$. Now, observe that at most $k-1$ colors are used on the neighbors of v, so there is at least one color in the list L_v that is not used on any of the neighbors of v. Hence, the coloring on $H-v$ can be extended to a proper coloring on H. This contradicts that H is not $\mathcal{L}$-choosable.

Thus, the minimum degree of H is at least k, and so H is a subgraph of the k-core of G. But the k-core of G is k-choosable by assumption, so H is also k-choosable, a contradiction. This completes the proof. □

The next result, called Brooks's theorem, gives a useful bound on the chromatic and choice numbers of a graph in terms of its maximum degree.

Theorem 3.4 (Brooks [23], Vizing [99], Erdős–Rubin–Taylor [40]). *Let G be a connected simple graph with maximum degree Δ. Then, G is Δ-colorable (resp. Δ-choosable), unless G is an odd cycle or the complete graph on $\Delta+1$ vertices, in which cases G is $(\Delta+1)$-chromatic (resp. $(\Delta+1)$-list-chromatic).*

Brooks proved his result for the chromatic number, and this was later extended to the choice number independently by Vizing and by Erdős, Rubin and Taylor.

Another useful result in computing the choice number of graphs is as follows.

Definition 3.5. For a directed graph G, a subgraph H is *Eulerian* if $\mathsf{indegree}_H(v) = \mathsf{outdegree}_H(v)$ for all $v \in V(H)$. An *even* (resp. *odd*) Eulerian subgraph is one with an even (resp. odd) number of edges.

Note that the subgraphs are not assumed to be connected in the above definition.

Theorem 3.6 (Alon–Tarsi [10]). *Suppose the edges of a graph G can be oriented such that the number of even Eulerian subgraphs differs from the number of odd Eulerian subgraphs. Let $\mathcal{L}$ be a list assignment such that $|L_v| \geq \mathsf{outdegree}(v)+1$ for all $v \in V(G)$. Then, G is $\mathcal{L}$-choosable.*

Notation. For every $m, r \geq 1$, denote by K_{m*r} the complete r-partite graph with m vertices in each part.

We will need the following bound on the growth of the choice number of K_{m*r}:

Theorem 3.7 (Alon [3]). *There exist two positive constants c_1 and c_2 such that, for every $m \geq 2$ and for every $r \geq 2$,*

$$c_1 r \log(m) \leq \chi_\ell(K_{m*r}) \leq c_2 r \log(m).$$

3.2 The gap for 6-regular triangulations on the torus

We start by observing that some of the graphs $T(r, s, t)$ are not simple triangulations: they may contain loops and multiple edges.

The graphs containing loops are precisely those isomorphic to $T(1, s, 0)$ for all $s \geq 1$, so we do not consider them. The loopless graphs containing multiple edges are precisely those isomorphic to $T(2, s, t)$ for $t = 0, s - 2, s - 1$, $s \geq 2$, and $T(1, s, t)$ for $t = 1, \lfloor (s - 1)/2 \rfloor$, $s \geq 3$. One can check that these graphs are never bipartite, so the chromatic number of any such graph is at least 3. Moreover, after deleting the duplicate edges (since they do not make a difference for the purpose of coloring) these graphs have maximum degree $\Delta \leq 5$. Hence, by Theorem 3.4, any such graph is 5-choosable and thus has gap at most 2, unless it is isomorphic to K_6; but in the latter case, Theorem 3.4 says that the chromatic number and choice number are both equal to 6, so the gap is 0. Hence, $\mathsf{jump}\big(T(r, s, t)\big) \leq 2$ if $T(r, s, t)$ contains multiple edges.

This takes care of the trivial cases. Next, we examine the graphs $T(r, s, t)$ that are simple. First, we establish some notations and definitions for use in the upcoming proofs:

Notation.

- Let G be a graph with a list assignment $\mathcal{L}$. For a subgraph G' of G and a color $c \in \mathbb{N}$, denote by $G'(c)$ the induced subgraph of G' on those vertices whose lists contain the color c. We shall denote (maximal connected) components of $G'(c)$ by α, β, etc.
- Let P be a path or cycle graph. A vertex $v \in V(P)$ is an *interior point* of P if $\mathsf{degree}(v) = 2$, and is otherwise an *end point* of P.
- Let P be a path or cycle graph, and let P' be a proper connected subgraph of P (so P' is itself a path). We denote by $v(P')$ an end point of P', and by $w(P')$ a vertex in $P \setminus P'$ that is adjacent to $v(P')$ (when it exists).

Next, we record the following simple observation:

Observation 3.8. Suppose P is a path or cycle of positive length, $\mathcal{L}$ is a list assignment on P of lists of size $k \geq 1$, and $c \in \mathbb{N}$ is any color such that $P(c)$ is a proper subgraph of P. Let α be a component of $P(c)$. Let $v(\alpha)$ be an end point of α for which $w(\alpha)$ exists. Then, there exists a color $d \in L_{w(\alpha)} \setminus L_{v(\alpha)}$, since $c \notin L_{w(\alpha)}$ and $|L_{v(\alpha)}| = |L_{w(\alpha)}| = k \geq 1$. In particular, there exists a component β of $P(d)$ containing $w(\alpha)$ but not $v(\alpha)$.

The following key lemma will allow us to color the alternate vertices of a path—equipped with a list assignment—in such a way that the remaining vertices lose at most one color from each of their lists.

Lemma 3.9. *Let $P = (V, E)$ be a path graph on at least 2 vertices, and $\mathcal{L}$ a list assignment on P of lists of size $k \geq 1$. Let $I_1 \cup I_2$ be the unique partition of $V(P)$ into two independent sets. Then, there exists a coloring of I_1 such that every $v \in I_2$ loses at most one color from its list.*

Here, by "losing a color from its list" we mean the following: if a vertex v has been colored with the color c, then we replace the list L_w with $L_w \setminus \{c\}$ for every vertex w which is a neighbor of v. So, if $c \in L_w$ for such a vertex w, then this vertex has *lost the color c from its list* due to the coloring on the vertex v.

Proof. Fix an end point $v(P)$ of P. Choose a color $c \in L_{v(P)}$ and consider the component α of $P(c)$ containing $v(P)$. Set $\mathsf{color}(v) = c$ for every $v \in I_1 \cap V(\alpha)$.

If $\alpha = P$, then we are done, since every $v \in I_2$ has lost at most one color from its list, namely c. So, suppose that $\alpha \neq P$, and define $P_1 := P - \alpha$. Then, P_1 is a path, and there is an end point $v(\alpha)$ of α such that $w(\alpha)$ is an end point of P_1. Choose $c_1 \in L_{w(\alpha)} \setminus L_{v(\alpha)}$, which exists by Observation 3.8. Let α_1 be the component of $P_1(c_1)$ containing $w(\alpha)$, and set $\mathsf{color}(v) = c_1$ for every $v \in I_1 \cap V(\alpha_1)$.

Again, if $\alpha_1 = P_1$, then we are done. Else, let $P_2 := P_1 \setminus \alpha_1$, and proceed inductively until the process terminates. □

3.2.1 Proof of Theorem 1.4

We restate the theorem for convenience:

Theorem 1.4. *For any loopless 6-regular triangulation G on the torus, $\mathsf{jump}(G) \leq 2$.*

Let us first show that any simple 6-regular toroidal triangulation G that is 3-chromatic is 5-choosable.

Theorem 3.10. *Let $G = T(r, s, t)$ be a simple 6-regular triangulation of the torus. If G is 3-chromatic, then G is 5-choosable.*

Proof. By Theorem 1.1, G is isomorphic to $T(r, s, t)$ for some integers $r \geq 1$, $s \geq 1$, and $0 \leq t < s$. It is straightforward to check that when $T(r, s, t)$ is a simple graph, it is 3-chromatic if and only if $s \equiv 0 \equiv r - t \pmod 3$. Moreover, $T(r, s, t)$ is uniquely 3-colorable whenever this happens. So, let $T(r, s, t)$ be 3-chromatic and let I_1, I_2, I_3 be the three independent sets defined by any 3-coloring of $T(r, s, t)$. Without loss of generality, we fix I_j to be the independent set containing the vertex $(1, j)$ for $j = 1, 2, 3$.

Consider the subgraph $G_1 := G - I_1$ of G. Note that G_1 is a 3-regular bipartite graph. Orient the edges of G_1 as follows: every horizontal edge is directed east, every vertical edge is directed north, and every diagonal edge is directed south-east. (Figure 3.1b shows this for $G = T(5,6,2)$.) More formally, give the orientations as follows (recall that addition in the second coordinate is taken modulo s). For every $1 \le i \le r$ and $1 \le j \le s$ such that $j - i \not\equiv 0 \pmod 3$:

- if $j-i \equiv 2 \pmod 3$, assign $(i,j) \to (i+1,j)$ for all $1 \le i < r$, and $(r,j) \to (1, j-t)$;
- if $j-i \equiv 1 \pmod 3$, assign $(i,j) \to (i,j+1)$ for all $1 \le i \le r$, $(i,j) \to (i+1, j-1)$ for all $1 \le i < r$, and $(r,j) \to (1, j-t-1)$.

Then, $\mathsf{outdegree}_{G_1}(v) = 2$ for every $v \in I_2$, and $\mathsf{outdegree}_{G_1}(v) = 1$ for every $v \in I_3$.

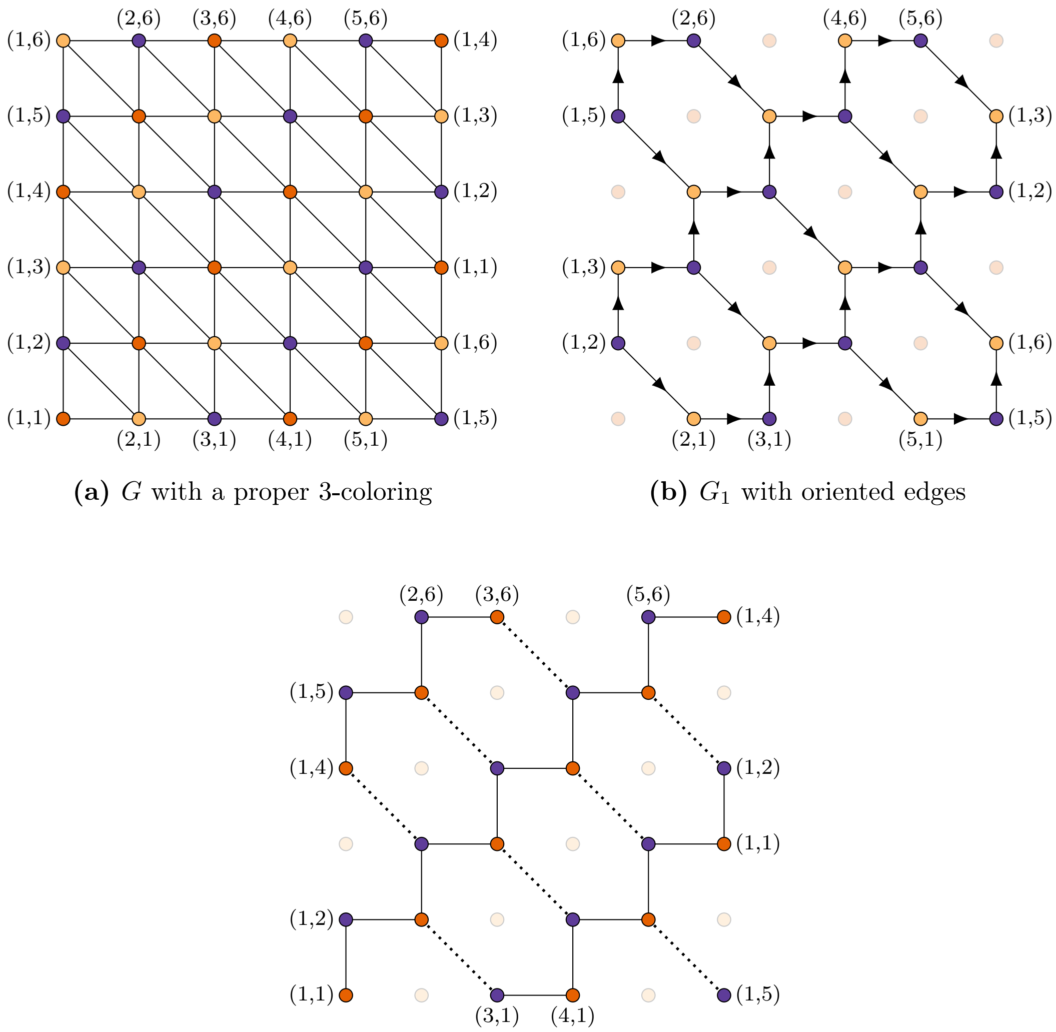
We claim that G_1 with this orientation has no odd Eulerian subgraphs. For, if H is an Eulerian subgraph of G_1 and $v \in V(H)$ is not an isolated vertex, then $\mathsf{degree}_H(v)$ is a positive even integer, so $\mathsf{degree}_H(v) = 2$. Thus, H is a disjoint union of cycles and isolated vertices, but every cycle in a bipartite graph is even, so this proves our claim. Since the empty graph is an even Eulerian subgraph of G_1, by Theorem 3.6 G_1 is $\mathcal{L}$-choosable for every list assignment $\mathcal{L}$ such that $|L_v| \ge 3$ for all $v \in I_2$ and $|L_v| \ge 2$ for all $v \in I_3$.

Now, suppose that we are given a 5-list assignment $\mathcal{L}$ on G. If we arbitrarily assign $\mathsf{color}(v) \in L_v$ for every $v \in I_1$, then—in the worst-case scenario—we are left with a 2-list assignment on $G_1 = G - I_1$. However, to apply Theorem 3.6 on G_1 with the orientation described above, we need 3-lists on I_2. So, let us now consider the subgraph H on $I_1 \cup I_2$ with only the horizontal and vertical edges present (see Figure 3.1c). We will show that we can color I_1 carefully in such a way that every vertex in I_2 loses at most one color in H; thus every vertex in I_2 will lose at most two colors in G (since every $v \in I_2$ is adjacent to one more vertex of I_1 in G than in H, shown by the dotted edges in Figure 3.1c). Then, we will be done by invoking Theorem 3.6.

Note that H is a disjoint union of even cycles (in fact, there are exactly $\gcd(s, r-t)/3$ cycles in H). First, assume for the sake of simplicity that there is only one cycle in H (as in Figure 3.1c). We eliminate the trivial case: if there is a color that belongs to every list on I_1, then we just color I_1 with that color, and we are done. Thus, we may assume that there is no color common to every list on I_1.

Now, suppose there exists a color c such that $H(c)$ has a component α of even order. Assign $\mathsf{color}(v) = c$ for all $v \in I_1 \cap V(\alpha)$. Observe that:

- Every $v \in V(\alpha) \cap I_2$ loses at most one color (namely, c) from its list.
- Since α has even order, it has endpoints $v_j(\alpha)$ in I_j for $j = 1, 2$.
- The vertex $w_1(\alpha) \in I_2$ has not lost any color in its list due to the coloring of $v_1(\alpha)$, so we are free to put any color on the other vertex adjacent to $w_1(\alpha)$.



(a) G with a proper 3-coloring

(b) G_1 with oriented edges

(c) The subgraph H

Figure 3.1: Illustration of the proof of Theorem 3.10 for the graph $G = T(5, 6, 2)$

- On the other hand, the vertex $w_2(\alpha) \in I_1$ must be colored carefully, since $v_2(\alpha) \in I_2$ has already lost one color from its list.

By Observation 3.8, there exists a color $d \in L_{w_2(\alpha)} \setminus L_{v_2(\alpha)}$. So, consider the path $P \coloneqq H - \alpha$. Apply Lemma 3.9 to P by starting the coloring at $w_2(\alpha)$ with the color d. Then, every $v \in I_2$ will indeed have lost at most one color from its list, so we are done.

On the other hand, suppose that for every color c, every component of $H(c)$ has odd order. Choose a color c and a component α of $H(c)$ such that both ends of α lie in I_1. We can always do this because if α' is a component of $H(c')$ such that both ends of α' lie in I_2, then choose $c \in L_{w(\alpha')} \setminus L_{v(\alpha')}$ for some end point $v(\alpha')$, and let α be the component of $H(c)$ containing $w(\alpha')$. Then, both ends of α lie in I_1 as required. Assign $\mathsf{color}(v) = c$ for all $v \in I_1 \cap V(\alpha)$. In this case, we have that:

- Every $v \in I_2 \cap V(\alpha)$ loses at most one color (namely, c) from its list.
- Since both ends, $v_1(\alpha)$ as well as $v_2(\alpha)$, lie in I_1, the vertices $w_1(\alpha)$ and $w_2(\alpha)$ in I_2 have not yet lost any color from their lists, so we are free to put any color on their other neighbors.

Thus, we consider the path $P \coloneqq H - \alpha$, and apply Lemma 3.9 to P starting from any end point with any color. Then, every $v \in I_2$ will again have lost at most one color from its list, so we are done.

If H consists of more than one cycle, we repeat this process for each cycle. Thus, with this coloring scheme for I_1, we are left with the required list sizes on the vertices of $G_1 = G - I_1$, so we are done by Theorem 3.6. Thus, we have shown that every simple 6-regular toroidal triangulation that is 3-chromatic is 5-choosable. □

Proof of Theorem 1.4. On the one hand, any simple triangulation requires at least 3 colors for a proper coloring. On the other hand, the torus has genus $g = 1$, and Heawood's formula says that $H(1) = 7$, so every toroidal graph requires no more than 7 colors for a proper coloring by Theorem 1.6. Furthermore, Theorem 1.7 shows that $\chi(G) = 7 \iff \chi_\ell(G) = 7$ for any toroidal graph G. Hence, if G is a simple triangulation on the torus with $\mathsf{jump}(G) > 2$, then it must be that $\chi(G) = 3$ and $\chi_\ell(G) = 6$. However, we have shown above that this cannot happen if G is 6-regular. Hence, $\mathsf{jump}(G) \leq 2$ for any simple 6-regular triangulation G on the torus. We have also seen that $\mathsf{jump}(G) \leq 2$ for any loopless 6-regular triangulation G that contains multiple edges. This completes the proof of Theorem 1.4. □

3.3 Asymptotics of the jump function

For the sake of simplicity, we ignore any ceilings and floors in the following proofs. We also restate the theorems for convenience.

3.3.1 Proof of Theorem 1.5

Theorem 1.5. $\mathsf{jump}(g) = \Theta(\sqrt{g})$. *That is, there exist positive constants* c_1 *and* c_2 *such that*

$$c_1\sqrt{g} \le \mathsf{jump}(g) \le c_2\sqrt{g}$$

for all sufficiently large g.

Proof. Let $g \ge 1$. For any graph G, $\chi_\ell(G) - \chi(G) \le \chi_\ell(G)$, and if G is embeddable in S_g then $\chi_\ell(G) \le H(g)$ by Theorem 1.6. Thus, $\mathsf{jump}(g) \le H(g) \le 7\sqrt{g}$ for all $g \ge 1$.

To establish the lower bound, let $m \ge 2$ be fixed, and consider the graph K_{m*r} for $r \ge 2$. Note that $\chi(K_{m*r}) = \chi(K_r) = r$, and $\chi_\ell(K_{m*r}) \ge c_1 r \log(m)$ by Theorem 3.7. Hence, $\mathsf{jump}(K_{m*r}) \ge c_1 r \log(m) - r$. Pick m large enough so that $c_1 \log(m) \ge 2$. Then, $\mathsf{jump}(K_{m*r}) \ge r$. Thus, we would like to show that K_{m*r} is embeddable in S_g for $g \le O(r^2)$.

Theorem 3.1 says that K_r is embeddable in $S_{\gamma(r)}$, so start with an embedding of K_r into $S_{\gamma(r)}$ and then add handles to $S_{\gamma(r)}$ to accommodate the extra edges of K_{m*r}. Since $|E(K_r)| = \binom{r}{2}$ and $|E(K_{m*r})| = m^2\binom{r}{2}$, we need to add at most $(m^2-1)\binom{r}{2}$ handles to $S_{\gamma(r)}$. Since $\gamma(r) + (m^2-1)\binom{r}{2} \le m^2r^2/2$, K_{m*r} is embeddable in $S_{g(r)}$ where $g(r) := m^2r^2/2$, and this is what we wanted to show. Hence, $\mathsf{jump}\big(g(r)\big) \ge r = c\sqrt{g(r)}$ for all $r \ge 2$, where $c := \sqrt{2}/m$.

Finally, suppose that $g(r) \le g' \le g(r+1)$ for a fixed $r \ge 2$. K_{m*r} is embeddable in $S_{g'}$ as well, so

$$\mathsf{jump}(g') \ge \mathsf{jump}(K_{m*r}) \ge r = c\sqrt{g(r+1)} - 1 \ge c\sqrt{g'} - 1 \ge c'\sqrt{g'}$$

for any positive constant $c' < c$, provided r (and hence $g(r)$) is sufficiently large. This completes the proof. Additionally, we note that the proof of Theorem 3.7 in [3] shows that one can take $c' = 2 \times 10^{-113}$. □

3.3.2 Proof of Theorem 1.9

Theorem 1.9. $\mathsf{jump}(g,r) = o(\sqrt{g})$ *when* $r = o\big(\sqrt{g}/\log(g)\big)$. *That is, if for each* $\delta > 0$ *we have* $r \le \delta\sqrt{g}/\log(g)$ *for all sufficiently large* g, *then for every* $\epsilon > 0$, $\mathsf{jump}(g,r) \le \epsilon\sqrt{g}$ *for all sufficiently large* g.

Proof. Let $\epsilon > 0$ and $g \geq 1$. Let G be simple and embeddable in S_g with $\chi(G) = r$, and let $|V(G)| = m$. To show that $\mathsf{jump}(G) \leq \epsilon\sqrt{g}$ it suffices to show that $\chi_\ell(G) \leq \epsilon\sqrt{g}$ since $\mathsf{jump}(G) \leq \chi_\ell(G)$. By Proposition 3.3, there is no loss of generality in assuming that G is equal to its $(\epsilon\sqrt{g})$-core. Thus, the minimum degree, and hence average degree $2\mathsf{e}/\mathsf{v}$, of G is bounded below by $\epsilon\sqrt{g}$. By the Euler–Poincaré formula, $2\mathsf{e}/\mathsf{v} \leq 6 + 12(g-1)/m$, and hence $m \leq 24\sqrt{g}/\epsilon$.

Now, G is a subgraph of K_{m*r}, and $\chi_\ell(G) \leq \chi_\ell(K_{m*r}) \leq c_2 r \log(m)$ by Theorem 3.7. Then, for every $\delta > 0$,

$$\begin{aligned}\chi_\ell(K_{m*r}) &\leq c_2 r \log(m) \\ &\leq c_2\left(\frac{\delta\sqrt{g}}{\log(g)}\right)\left(\log(24/\epsilon) + \frac{1}{2}\log(g)\right) \\ &\leq c_2\delta\sqrt{g}\left(\frac{\log(24/\epsilon)}{\log(g)} + \frac{1}{2}\right) \\ &< c_2\delta\sqrt{g}\end{aligned}$$

for all sufficiently large g. Let $\delta = \epsilon/c_2$ to finish the proof. □

3.4 Conclusions and further remarks

3.4.1 The jump for general toroidal graphs

We discuss some partial results towards computing $\mathsf{jump}(G)$ for a general toroidal graph G.

Firstly, we will need the following result of Alon and Tarsi [10] used in their proof that bipartite planar graphs are 3-choosable. For a graph G, define

$$L(G) := \max_{H \leq G}\left\{\frac{|E(H)|}{|V(H)|}\right\},$$

where the maximum is taken over all subgraphs $H = (V, E)$ of G.

Theorem 3.11 (Alon–Tarsi [10]). *Every bipartite graph G is $(\lceil L(G)\rceil + 1)$-choosable.*

Using the Euler–Poincaré formula, it follows that any planar bipartite graph $G = (V, E)$ satisfies $\mathsf{e} \leq 2\mathsf{v} - 4$, so $L(G) \leq 2$. Hence, every bipartite planar graph is 3-choosable.

The same analysis goes through for toroidal bipartite graphs, since the Euler–Poincaré formula applied on toroidal bipartite graphs $G = (V, E)$ yields $\mathsf{e} \leq 2\mathsf{v}$, so again $L(G) \leq 2$. Thus, every bipartite toroidal graph is also 3-choosable. Hence, if G is a toroidal graph with $\mathsf{jump}(G) > 2$, then G cannot be bipartite, so it must be that $\chi(G) \geq 3$. On the other

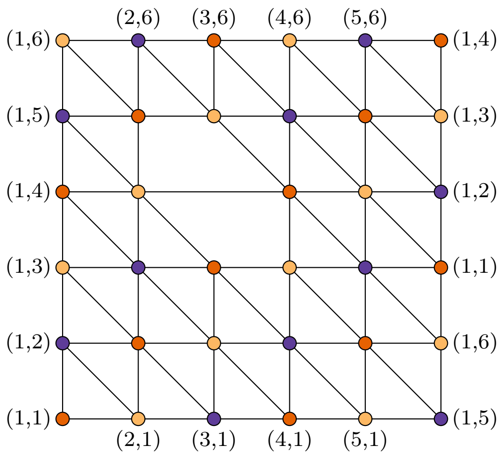


Figure 3.2: Toroidal G with $\chi(G) = 3$, $\delta(G) = 5$, $\Delta(G) = 6$

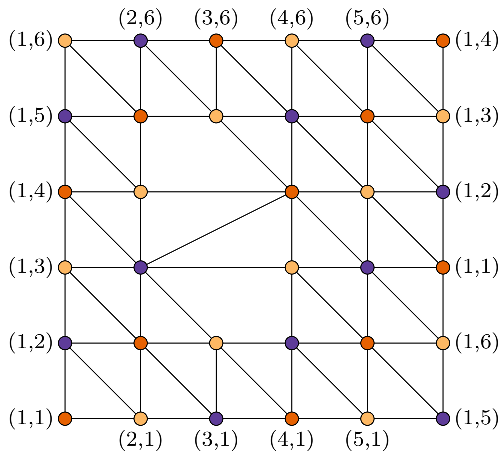


Figure 3.3: Toroidal G with $\chi(G) = 3$, $\delta(G) = 5$, $\Delta(G) = 7$

hand, if $\chi(G) \geq 4$, then $\mathsf{jump}(G) \leq 2$, since every toroidal graph is 7-choosable, but every 7-list chromatic toroidal graph is also 7-chromatic, by Theorem 1.7.

Hence, any counterexample to the claim that $\mathsf{jump}(1) = 2$ must be a toroidal graph G for which $\chi(G) = 3$ and $\chi_\ell(G) = 6$. Now, suppose that there do exist such graphs, and choose one with the minimal number of vertices. Then, by Proposition 3.3, its minimum degree is at least 5. Recall that the Euler–Poincaré formula shows that the average degree, $2\mathsf{e}/\mathsf{v}$, of a toroidal graph G satisfies $2\mathsf{e}/\mathsf{v} \leq 6$, and equality holds if G is a triangulation. Thus, if $\delta(G) = 6$, then G is in fact a 6-regular triangulation. This motivates one to examine 6-regular triangulations in particular, and we have shown in Theorem 1.4 that $\mathsf{jump}(G) \leq 2$ for such graphs.

Thus, any minimal counterexample G must satisfy $\delta(G) = 5$. Moreover, if we add edges to G while preserving its toroidicity and chromatic number, then we can get a minimal counterexample whose faces are all either triangular or quadrangular, which we call a *mosaic* following Nakamoto, Noguchi and Ozeki [73]. Thus, it follows that $\mathsf{jump}(1) > 2$ if and only if there is a 3-chromatic toroidal mosaic of minimum degree 5 that is not 5-choosable.

Now, it so happens that there do exist 3-chromatic toroidal graphs G with $\delta(G) = 5$ (see Figures 3.2, 3.3, and 3.4; again, the edges between the top and bottom rows are omitted). While ad hoc arguments can show that $\mathsf{jump}(G) \leq 2$ for some of these graphs, a unified argument is still missing. An Alon–Tarsi type argument appears more difficult to implement since there may be several vertices with degree greater than 10, so it is not clear how to orient the edges appropriately.

A relevant concept to mention here is *color criticality* of graphs. For $k \geq 1$, a *k-critical* graph is one that is not $(k-1)$-colorable but whose proper subgraphs are. Similarly, a

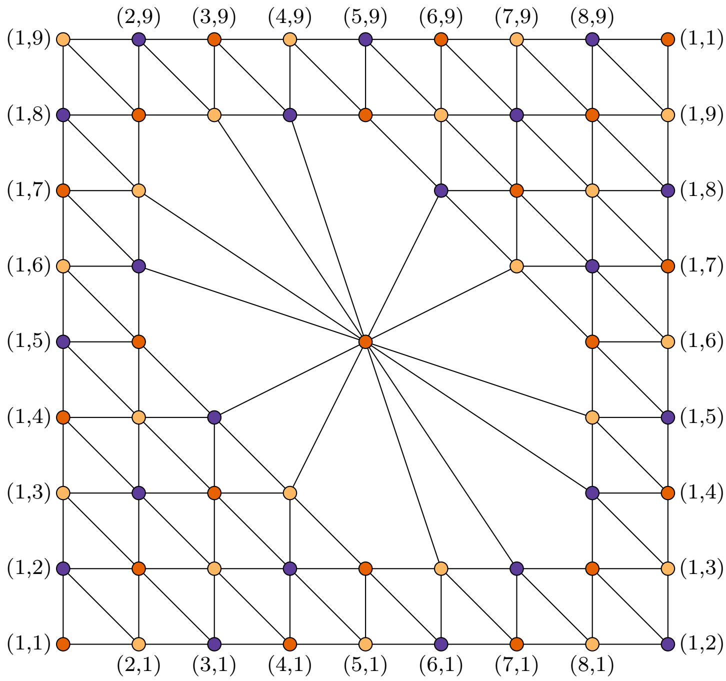


Figure 3.4: Toroidal G with $\chi(G) = 3$, $\delta(G) = 5$, $\Delta(G) = 12$

k-list-critical graph is one for which there is a k-list assignment $\mathcal{L}$ such that the graph is not $\mathcal{L}$-choosable but every proper subgraph is. Every k-critical graph is thus also k-list-critical, but in general there are k-list-critical graphs that are not k-critical. Furthermore, a k-critical graph cannot contain another k-critical graph as a proper subgraph, but this statement is not true if we replace "k-critical" with "k-list-critical", since a graph may be list-critical with respect to a list assignment $\mathcal{L}$ but not $\mathcal{L}'$, yet it may contain a proper subgraph that is list-critical with respect to $\mathcal{L}'$ but not $\mathcal{L}$. So, one defines a *minimal* k-list-critical graph to be one which does not contain a k-list-critical graph as a proper subgraph.

Observe that a minimal counterexample to our claim that also has the least number of edges must be a minimal 6-list-critical graph. Postle and Thomas [80] showed that there are only finitely many 6-list-critical graphs on any surface. Thus, the claim that every toroidal graph has gap at most 2 needs to be verified against only finitely many potential counterexamples; however, a complete list of 6-list-critical graphs on the torus does not yet exist in the literature (though a complete list of 6-critical graphs was given by Thomassen [96]). Stiebitz, Tuza and Voigt [92] showed that for all $2 \leq r \leq k$, there is a minimal k-list critical graph that is r-chromatic, so one also cannot immediately rule out the existence of a minimal 6-list-critical graph on the torus that is 3-chromatic.

We note that computing tight bounds even for triangulations on surfaces of higher values of g appears difficult due to the lack of structural results similar to Altshuler's theorem for the torus (Theorem 1.1).

3.4.2 Analogous results for graphs embeddable in nonorientable surfaces

As mentioned in Section 1.2.1, the restriction to orientable surfaces has only been for convenience, and the results proved in Section 3.3 also hold over nonorientable surfaces with suitable modifications, which we indicate below.

The Heawood number for nonorientable surfaces is

$$\widetilde{H}(k) := \left\lfloor \frac{7 + \sqrt{1 + 24k}}{2} \right\rfloor.$$

By essentially the same argument as given by Heawood [52] for the orientable case, $\widetilde{H}(k)$ is an upper bound for the chromatic number of any graph G embeddable in N_k, $k \geq 1$, and the argument also carries forward to the choice number, so $\chi(G) \leq \chi_\ell(G) \leq \widetilde{H}(k)$ for every G embeddable in N_k, $k \geq 1$.

This upper bound is not tight for the Klein bottle N_2, as shown by Franklin [45]: $\widetilde{H}(2) = 7$, but every graph embeddable in N_2 is 6-colorable. In particular, K_6 is embeddable in N_2, but K_7 is not. Using Theorem 3.4 one can show that every graph embeddable in N_2 is also 6-choosable.

For every other nonorientable surface, Heawood's upper bound is tight. Ringel [85] showed that for every $r \geq 1$, except $r = 2$, K_r is embeddable in $N_{\tilde{\gamma}(r)}$ for $\tilde{\gamma}(r) := \lceil (r-3)(r-4)/6 \rceil$ and this is best possible. This implies that $K_{\widetilde{H}(k)}$ is embeddable in N_k for all $k \geq 1$ except $k = 2$. Dirac's map color theorem [1, 35] also extends to N_k for all $k \geq 1$, except $k = 2$. In the latter case, K_6 is not the only 6-chromatic graph that is embeddable in N_2 (see [1] for an example of another such graph). Dirac's map color theorem for the choice number [22, 63] also extends to N_k for all $k \geq 1$, except $k = 2$.

We define, analogously, the functions $\widetilde{\mathsf{jump}}(k)$ and $\widetilde{\mathsf{jump}}(k, r)$. Using the above results it is not hard to show the following:

Theorem 3.12. $\widetilde{\mathsf{jump}}(1) = 2$. *That is, for graphs G embeddable in the projective plane N_1, $\mathsf{jump}(G) \leq 2$. Moreover, this bound is best possible.*

The asymptotic bounds on $\widetilde{\mathsf{jump}}(k)$ and $\widetilde{\mathsf{jump}}(k, r)$ are also the same as in the orientable case. The proofs go through in a similar manner. One just needs to know what the result is of attaching a handle to a nonorientable surface, for which the following theorem is useful:

Theorem 3.13 (Dyck [36]). *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

One also needs to modify the Euler–Poincaré formula for nonorientable surfaces as follows: if G is embeddable in N_k, $k \geq 1$, and v, e and f denote the number of vertices, edges, and faces of G in an embedding of G in N_k, respectively, then $\mathsf{v} - \mathsf{e} + \mathsf{f} \geq 2 - k$, with equality holding if every face is homeomorphic to a disc.

3.4.3 Concluding remarks

We have proved in Theorem 1.4 that any loopless 6-regular triangulation G on the torus satisfies $\mathsf{jump}(G) \leq 2$. We speculate that *every* 3-chromatic toroidal graph is in fact 5-choosable:

Conjecture 3.14. $\mathsf{jump}(1, 3) = 2$.

A resolution to Conjecture 3.14 will also answer Question 1.3, as we have observed that the maximum gap for toroidal graphs is at least 2, and neither toroidal bipartite graphs nor toroidal graphs with chromatic number at least 4 can attain a gap greater than 2.

An easy example of a 6-regular triangulation G on the torus for which $\mathsf{jump}(G) = 1$ is furnished by $T(3, 3, 0)$, which is isomorphic to the complete multipartite graph K_{3*3}. Kierstead [59] showed that $\chi_\ell(K_{3*r}) = \lceil (4r - 1)/3 \rceil$ for every $r \geq 1$, so $\chi_\ell(K_{3*3}) = 4$, and since $T(3, 3, 0) \equiv T(r, s, t)$ satisfies $s \equiv 0 \equiv r - t \pmod 3$, it is 3-chromatic. Hence, $\mathsf{jump}\big(T(3, 3, 0)\big) = 1$.

We are also able to show that if G is any 3-chromatic 6-regular toroidal triangulation, then G is not 3-choosable (this is discussed in the next chapter). But, we do not have an explicit example of a 6-regular toroidal triangulation for which $\mathsf{jump}(G) = 2$. So, we pose the following question:

Question 3.15. *Are there 3-chromatic 6-regular toroidal triangulations that are 5-list-chromatic?*

The results on the asymptotic behavior of $\mathsf{jump}(g)$ and $\mathsf{jump}(g, r)$ motivate the following conjectures that refine the results proved in Section 3.3:

Conjecture 3.16. $\mathsf{jump}(g, r)$ *is unimodal in r for each fixed g. That is, there exists $r_0 \equiv r_0(g)$ such that*

$$\mathsf{jump}(g, 1) \leq \mathsf{jump}(g, 2) \leq \cdots \leq \mathsf{jump}(g, r_0) \geq \cdots \geq \mathsf{jump}\big(g, H(g)\big).$$

This is already seen to be true for planar and toroidal graphs. Firstly, a loopless graph is 1-chromatic if and only if it is empty, which implies that it is also 1-list-chromatic, so $\mathsf{jump}(g, 1) = 0$ for all $g \geq 0$. Next, the results mentioned in Section 1.2.1 show that $\mathsf{jump}(0, 2) = 1$, $\mathsf{jump}(0, 3) = 2$, and $\mathsf{jump}(0, 4) = 1$. Lastly, $\mathsf{jump}(0, r) = 0$ for $r \geq 5$ by the four color theorem due to Appel and Haken [13, 14]. So, $\mathsf{jump}(0,r)$ is indeed unimodal in r.

Similarly, for toroidal graphs, though we don't have precise values of $\mathsf{jump}(g,r)$ for all r, the discussion so far shows that $\mathsf{jump}(1,2)=1$, $\mathsf{jump}(1,3)=2$ or 3, $\mathsf{jump}(1,4)=1$ or 2, $\mathsf{jump}(1,5)=0$ or 1, $\mathsf{jump}(1,6)=0$ and $\mathsf{jump}(1,7)=0$. So, $\mathsf{jump}(1,r)$ is unimodal in r as well.

It is not hard to show that unimodality also holds for $\widetilde{\mathsf{jump}}(1,r)$ and $\widetilde{\mathsf{jump}}(2,r)$, that is, for the projective plane and the Klein bottle; so, Conjecture 3.16 can be extended to the nonorientable case as well.

One also notices that there may be more than one value r_0 at which the maximum gap is attained. So, define $r_{\max} \equiv r_{\max}(g)$ to be the least value of $r_0(g)$ in Conjecture 3.16. We conjecture that $\sqrt{g}$ is the correct order of r at which $\mathsf{jump}(g,r)$ attains its maximum value for each fixed g:

Conjecture 3.17. *For all sufficiently large g, $r_{\max} = \Theta(\sqrt{g})$.*

Again, Conjecture 3.17 can be extended analogously to the nonorientable case as well.

Finally, a structural result on the Klein bottle N_2 for 6-regular triangulations, similar in spirit to Altshuler's theorem on the torus (Theorem 1.1), was given independently by Negami [75] and Thomassen [95]. We pose the following question on the Klein bottle analogous to Question 1.3 for the torus, and also ask whether the 6-regular triangulations on the Klein bottle can be examined to get a result analogous to Theorem 1.10.

Question 3.18. *What is $\widetilde{\mathsf{jump}}(1)$? That is, how large can the gap between the choice number and chromatic number be for a graph embeddable on the Klein bottle?*

Just as in the toroidal case, it is not hard to show that the maximum gap cannot be smaller than 2, and we ask how large this gap can get.

Question 3.19. *What is the maximum value of $\widetilde{\mathsf{jump}}(G)$ for any loopless 6-regular triangulation G on the Klein bottle?*

However, computing $\widetilde{\mathsf{jump}}(k)$ precisely for higher values of k seems difficult for the same reason as in the orientable case.

Lastly, the proof of Theorem 1.5 shows that

$$c_1\sqrt{g} \le \mathsf{jump}(g) \le c_2\sqrt{g}$$

for all sufficiently large g, with $c_1 = 2\times 10^{-113}$ and $c_2 = 7$. It would be interesting to see if this can be refined further:

Question 3.20. *Does the limit*

$$\lim_{g\to\infty} \frac{\mathsf{jump}(g)}{\sqrt{g}}$$

exist? If yes, what is its value?

Of course, one can also raise the analogous question for $\widetilde{\mathsf{jump}}(k)$ as well.

Chapter 4

List coloring the 6-regular toroidal triangulations

In the previous chapter, we showed that every 3-chromatic 6-regular triangulation of the torus is 5-choosable, by applying the Alon–Tarsi theorem. As discussed in the introduction, a different method will be required to give an efficient algorithm for 5-list coloring these graphs. The linear time algorithm that we describe in this chapter will be effective on a large class of 6-regular triangulations of the torus, and in particular on the 3-chromatic ones.

4.1 Preliminaries

We shall use the notation C_i, for $1 \leq i \leq r$, to denote the induced subgraph of $T(r, s, t)$ on the ith column of $T(r, s, t)$, that is, on the set of vertices $\{(i, j) : 1 \leq j \leq s\}$. Note that each C_i is a cycle of length s.

The following lemma is useful in simplifying arguments through the use of symmetry:

Lemma 4.1. *Let $r \geq 1$, $s \geq 1$ and $0 \leq t \leq s-1$. The map $(i, j) \mapsto (r-i+1, s-j+1)$ on $V\big(T(r, s, t)\big)$ induces an automorphism of $T(r, s, t)$.*

In particular, this automorphism reverses the ordering of the rows (as well as of the columns).

For integers $r, s \geq 3$, define the *cylindrical triangulation* $C(r, s)$ to be the graph obtained from $T(r+1, s, 0)$ by deleting the column C_{r+1}. More formally, let $V\big(C(r, s)\big) := \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and let $E\big(C(r, s)\big)$ contain the following edges:

- For $1 < i < r$, let (i, j) be adjacent to $(i, j \pm 1)$, $(i \pm 1, j)$ and $(i \pm 1, j \mp 1)$.
- Let $(1, j)$ be adjacent to $(1, j \pm 1)$, $(2, j)$ and $(2, j-1)$.

- Let (r, j) be adjacent to $(r, j \pm 1)$, $(r-1, j+1)$ and $(r-1, j)$.

Again, addition in the second coordinate is taken modulo s. Note that every *interior vertex* of $C(r, s)$, that is, any vertex (i, j) with $1 < j < r$, has degree 6 and every *exterior vertex* of $C(r, s)$, that is, any vertex (i, j) with $j = 1$ or $j = r$, has degree 4.

By an abuse of notation, we shall use C_i to denote the induced subgraph on the ith column of $C(r, s)$, too. Note that if we delete any column of the graph $T(r+1, s, t)$ for any $0 \leq t \leq s-1$, we still get a graph isomorphic to $C(r, s)$.

We will need the following theorem on finding matchings in regular bipartite graphs.

Definition 4.2. A *matching* in a graph $G = (V, E)$ is a subset M of E such that no two edges in M have a common vertex. A matching is said to be *perfect* if every vertex $v \in V$ belongs to some edge in the matching.

A standard application of Hall's marriage theorem shows that any regular bipartite graph has a perfect matching (see [103, Corollary 3.1.13]). The following result says that we can also efficiently find such a perfect matching.

Theorem 4.3 (Cole–Ost–Schirra [31], 2001)**.** *Let $G = (V, E)$ be a regular bipartite graph. Then, a perfect matching M of G can be found in $O(|E|)$ time.*

We make the following definition which will simplify some of the terminology in the proofs that follow.

Definition 4.4. Let $\mathcal{L}$ be a list assignment on a graph $G = (V, E)$. If, in a partial $\mathcal{L}$-coloring of G, a vertex $v \in V$ is colored with $c \in L_v$, then the color c is no longer available for use on the uncolored neighbors of v. So, the color c is removed from the lists of the neighbors of v, and we do this for each vertex colored in this partial $\mathcal{L}$-coloring of G. The new lists on G are also called the *residual* lists on G, and we shall say that the list on an uncolored vertex u *reduces by* k if the residual list on u is a $(|L_u| - k)$-list.

The following well-known lemma, due to Bondy–Bopanna–Siegel [10, Remark 2.4], gives an algorithmic proof of Theorem 3.6 in the case when G is given an orientation containing no odd directed cycle. For the sake of completeness, we provide a proof of this lemma along the lines in [103].

Definition 4.5. Let G be a digraph, and $v \to w$ be an edge of G. We also call w the *successor* of v, and v the *predecessor* of w. A *kernel* of G is an independent set S such that every $v \notin S$ has a successor in S.

Lemma 4.6 (Bondy–Bopanna–Siegel [10, Remark 2.4], 1992)**.** *Let $G = (V, E)$ be a simple graph that is given an orientation containing no odd directed cycle. Suppose $\mathcal{L}$ is a list* *assignment on G such that L_v is an $(\mathsf{outdegree}(v)+1)$-list for all $v \in V$. Then, G is $\mathcal{L}$-choosable.*

Proof. If $|V| = 1$, then the statement is trivial, so suppose that $|V| = n > 1$, and that the lemma is true for all graphs with fewer than n vertices. Let c be a color that occurs in some list assigned by $\mathcal{L}$. Consider the induced subgraph H on the set $U := \{v \in V : c \in L_v\}$. Clearly, the induced orientation on H also does not have any odd directed cycle. Now, Richardson's theorem [84] says that any digraph without odd directed cycles has a kernel, so let S be a kernel in H. Assign the color c to every vertex in S, and now consider $G' := G - S$. Notice that the residual lists have reduced in size by 1 for every list on $U - S$, but every vertex in $U - S$ also has a successor in S. Thus, G' satisfies the induction hypothesis, and we are done. □

One can find in the literature [76, 93] proofs of Richardson's theorem that output a kernel in polynomial time. However, the graphs that we consider (cf. Lemmas 4.7 and 4.15) have enough structure that they permit straightforward linear time algorithms for finding a kernel.

We will often need to color paths and cycles in $T(r, s, t)$ and $C(r, s)$, so we compile well-known results (see [40], for instance) on the colorability of these graphs in the following lemma. Moreover, from Lemma 4.6 and the above comments, we can give linear time algorithms for $\mathcal{L}$-coloring these graphs.

Lemma 4.7.

1. *An even cycle is 2-list chromatic.*
2. *An odd cycle is not 2-colorable, and hence not 2-choosable. However, if $\mathcal{L}$ is a list assignment of 2-lists on an odd cycle such that not all the lists are identical, then the cycle is $\mathcal{L}$-choosable.*
3. *If $\mathcal{L}$ is a list assignment on an odd cycle having one 1-list, one 3-list, and all the rest as 2-lists, then the cycle is $\mathcal{L}$-choosable.*
4. *If $\mathcal{L}$ is a list assignment on a path graph having one 1-list, and all the rest as 2-lists, then the path is $\mathcal{L}$-choosable.*

Moreover, the $\mathcal{L}$-colorings can all be found in linear time.

The following lemma, due to S. Sinha (during an undergraduate research internship with N. Balachandran), is in a similar spirit to Thomassen's list coloring of a near-triangulation of the plane [97], and it will be repeatedly invoked in the proof of case (1) in Theorem 1.10.

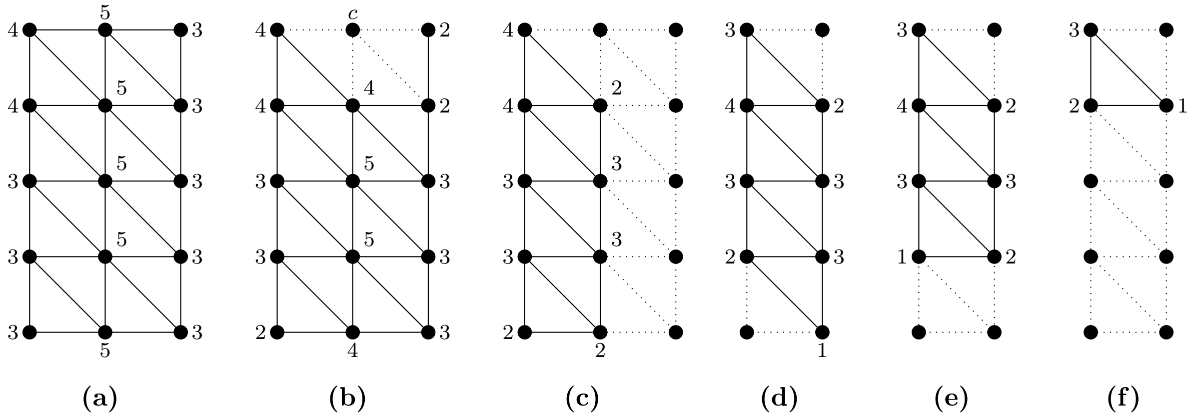


Figure 4.1: Illustration of the sizes of the lists on the vertices at each step for $G = C(3,5)$

Lemma 4.8 (Sinha, 2014)**.** *For $r \geq 3$, $s \geq 3$, let $G = C(r,s)$ be a cylindrical triangulation. Suppose that $\mathcal{L}$ is a list assignment on G such that:*

1. *there exists $1 \leq j \leq s$ such that the exterior vertices $(1,j)$ and $(1,j-1)$ have lists of size equal to 4;*
2. *every other exterior vertex has a list of size equal to 3;*
3. *every interior vertex has a list of size equal to 5.*

Then, G is $\mathcal{L}$-choosable. Moreover, an $\mathcal{L}$-coloring can be found in linear time.

Proof. By Lemma 4.7, there is a proper coloring of C_r since it is assigned 3-lists under $\mathcal{L}$. Since every vertex of C_{r-1} is adjacent to exactly two vertices of C_r, a proper coloring of C_r reduces the 5-lists on C_{r-1} to 3-lists.

Thus, by inductively coloring the columns of $C(r,s)$ from the right, we may assume without loss of generality that $r = 3$. We also assume without loss of generality that the lists of size equal to 4 are on the vertices $(1, s-1)$ and $(1, s)$ in the column C_1. Now, color $(2, s)$ with $c \in L_{(2,s)} \setminus L_{(1,s)}$, which exists since C_2 has 5-lists. This reduces the sizes of the lists on each of the neighbors of $(2, s)$ by 1, except for $L_{(1,s)}$, which still has size equal to 4. Now, C_3 has 3-lists on every vertex, except for $(3, s)$ and $(3, s-1)$, which have 2-lists. So, properly color C_3 using Lemma 4.7. Then, color the remaining vertices in a zigzag fashion from the bottom row, coloring $(1, s)$ last, in the following order: $(1, 1)$, $(2, 1)$, $(1, 2)$, $(2, 2)$, $\ldots$, $(1, j)$, $(2, j)$, $\ldots$, $(1, s-2)$, $(2, s-2)$, $(2, s-1)$, $(1, s-1)$, $(1, s)$.

A proper coloring can always be found by coloring the vertices in the above sequence for the following reason. After coloring C_3, the list sizes on the remaining vertices are as follows: $(1, s)$ and $(1, s-1)$ have 4-lists, $(1, 1)$, $(2, 1)$ and $(2, s-1)$ have 2-lists, and all other vertices have 3-lists. So, color the vertex $(1, 1)$ using a color from its list, and the list sizes then are as follows: $(1, s-1)$ has a 4-list, $(1, 2)$ and $(2, s-1)$ have 2-lists, $(2, 1)$

has a 1-list, and all other vertices have 3-lists. Next, color $(2,1)$ using a color from its list, and observe that the next vertex that is to be colored in the sequence always has at least one color left in its list. The last three vertices left to be colored are in a 3-cycle, with lists of sizes at least 1, 2 and 3. This cycle is properly colorable by Lemma 4.7, so this completes the proof.

It is clear from the proof that this algorithm produces an $\mathcal{L}$-coloring in linear time. Fig. 4.1 illustrates the sizes of the lists at each step of the above coloring sequence for the graph $G = C(3,5)$. $\square$

4.2 Preparation for the proof of case (1) in Theorem 1.10

Recall that case (1) in Theorem 1.10 says that if G is a simple graph isomorphic to $T(r,s,t)$ for $r \geq 4$, then G is 5-choosable. So, for $r \geq 4$, $s \geq 3$ and $0 \leq t \leq s-1$, let $G := T(r,s,t)$. Fix $\mathcal{L}$ to be a list assignment on G of lists of size equal to 5. We start be eliminating the trivial case: if all the lists of $\mathcal{L}$ are identical, then G is $\mathcal{L}$-choosable because G is 5-colorable [96]. Moreover, a 5-coloring can be found in linear time: see [32, 88, 104].

For a vertex $(i,j) \in C_i$, let its *left neighbors* be the two adjacent vertices in C_{i-1}, its *right neighbors* be the two adjacent vertices in C_{i+1}, and its *vertical neighbors* be the two adjacent vertices in C_i. We shall repeatedly invoke Lemma 4.8 to cut down on the possible choices for the lists assigned by $\mathcal{L}$, until it becomes simple enough to directly specify a proper coloring.

Lemma 4.9. *Suppose that not all the lists in $\mathcal{L}$ are identical. If there is a vertex $v \in V(G)$ such that its list is not contained in the union of the lists of its two left neighbors, then G is $\mathcal{L}$-choosable in linear time.*

Proof. Choose a color for v that is not in the list of either left neighbor of v, and extend the coloring to the cycle C_i containing v by Lemma 4.7. Then, we are left to color a graph isomorphic to $C(r-1,s)$ equipped with lists whose sizes satisfy the hypotheses of Lemma 4.8. Hence, the coloring on C_i extends to a proper coloring of G in linear time by Lemma 4.8, and so we are done. $\square$

Note that by Lemma 4.1 the above lemma is also true when "left neighbors" is replaced by "right neighbors" in the statement. Thus, it suffices to assume that the list assignment $\mathcal{L}$ satisfies the following criterion:

(i) Not all the lists in $\mathcal{L}$ are identical, and for every vertex $(i,j) \in V(G)$, $L_{(i,j)} \subseteq L_{(i-1,j)} \cup L_{(i-1,j+1)}$ and $L_{(i,j)} \subseteq L_{(i+1,j)} \cup L_{(i+1,j-1)}$.

In particular, we may assume that no column has identical lists, for if C_i has identical lists, then so do C_{i-1} and C_{i+1} by criterion (i), so all the lists in $\mathcal{L}$ are identical by induction, a contradiction.

Lemma 4.10. *Suppose that $\mathcal{L}$ satisfies criterion (i).*

(1) Let $(i,j),(i,j-1) \in V(G)$ have distinct lists. Suppose one of the following conditions holds:

(a) $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} \neq L_{(i-1,j-1)}$;

(b) $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} \neq L_{(i-1,j)}$;

(c) $L_{(i,j)} \neq L_{(i-1,j)}$ and $L_{(i,j-1)} \neq L_{(i-1,j-1)}$.

Then, G is $\mathcal{L}$-choosable in linear time.

(2) Suppose $u,v \in V(G)$ are adjacent vertices lying on distinct columns such that $L_u = L_v$. If for every vertex $w \in V(G)$ that is adjacent to both u and v we have $L_w \neq L_u$, then G is $\mathcal{L}$-choosable in linear time.

(3) Let $(i,j) \in V(G)$ be a vertex such that both its left neighbors have lists identical to $L_{(i,j)}$. Suppose that $L_{(i,j)} \neq L_{(i,j+1)}$ and $L_{(i,j)} \neq L_{(i,j-1)}$. Then, G is $\mathcal{L}$-choosable in linear time.

Proof.

(1) (a) Choose a color for $(i-1,j+1)$ from $L_{(i-1,j+1)} \setminus L_{(i,j)}$, for $(i-1,j-1)$ from $L_{(i-1,j-1)} \setminus L_{(i,j-1)}$, and extend this to a proper coloring of C_{i-1} by Lemma 4.7. Then, we are in the scenario of Lemma 4.8, and so we are done.

(b) Choose a color c for $(i-1,j)$ from $L_{(i-1,j)} \setminus L_{(i,j-1)}$. By criterion (i), $c \in L_{(i,j)}$, so there exists a color d $(\neq c) \in L_{(i-1,j+1)} \setminus L_{(i,j)}$. Color $(i-1,j+1)$ with d and extend this to a proper coloring of C_{i-1} by Lemma 4.7. Then, we are in the scenario of Lemma 4.8, and so we are done.

(c) This is similar to the preceding case. Choose a color c for $(i-1,j)$ from $L_{(i-1,j)} \setminus L_{(i,j)}$. By criterion (i), $c \in L_{(i,j-1)}$, so there exists a color d $(\neq c) \in L_{(i-1,j-1)} \setminus L_{(i,j-1)}$. Color $(i-1,j-1)$ with d and extend this to a proper coloring of C_{i-1} by Lemma 4.7. Then, we are in the scenario of Lemma 4.8, and so we are done.

(2) Suppose $u=(i,j)$ and $v=(i-1,j)$, Then, neither $(i-1,j+1)$ nor $(i,j-1)$ has a list identical to L_u. But then we are in the scenario of Lemma 4.10(1)b, so G is $\mathcal{L}$-choosable in linear time. Similarly, let $u=(i,j-1)$ and $w=(i-1,j)$, Then,

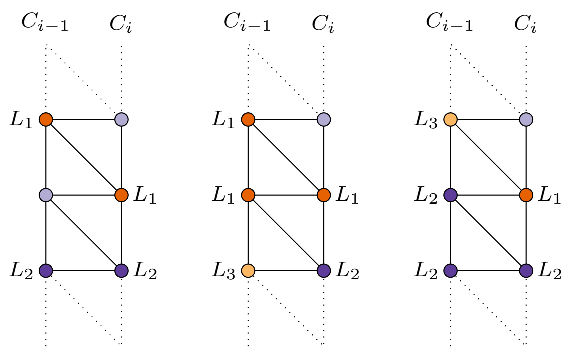


Figure 4.2: Illustrations of configurations (a) to (c) in criterion (ii)

neither $(i-1, j-1)$ nor (i, j) has a list identical to L_u. But then we are in the scenario of Lemma 4.10(1)c, so again G is $\mathcal{L}$-choosable in linear time.

(3) Choose a color for $(i, j+1)$ from $L_{(i,j+1)} \setminus L_{(i-1,j+1)}$, for $(i, j-1)$ from $L_{(i,j-1)} \setminus L_{(i-1,j)}$, and extend this to a proper coloring of C_i by Lemma 4.7. Then, we are in the scenario of Lemma 4.8, so we are done. □

Note that, by Lemma 4.1, Lemma 4.10(1) is also true when the list assignment $\mathcal{L}$ instead satisfies one of three analogous conditions relating the lists on (i, j) and $(i, j-1)$ with their right neighbors, and Lemma 4.10(3) is also true when "left neighbors" is replaced by "right neighbors" in the statement.

Thus, in addition to criterion (i), we may also assume the following criteria:

(ii) Whenever (i, j) and $(i, j-1)$ have distinct lists assigned by $\mathcal{L}$, one of the following three configurations holds:

   (a) $L_{(i,j)} = L_{(i-1,j+1)}$ and $L_{(i,j-1)} = L_{(i-1,j-1)}$;

   (b) $L_{(i,j)} = L_{(i-1,j+1)} = L_{(i-1,j)}$ and $L_{(i,j-1)} \neq L_{(i-1,j-1)}$;

   (c) $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} = L_{(i-1,j)} = L_{(i-1,j-1)}$.

(iii) whenever u and v are adjacent vertices on distinct columns with $L_u = L_v$, there is a vertex w adjacent to both u and v such that $L_w = L_u = L_v$.

(iv) whenever u, v and w are mutually adjacent vertices having identical lists, with v and w lying on the same column, at least one of the vertical neighbors of u has a list identical to L_u.

To see how we arrive at these criteria, suppose that none of the scenarios in Lemma 4.10(1) hold for the list assignment $\mathcal{L}$. In particular, since Lemma 4.10(1)a doesn't hold, we must have $L_{(i,j)} = L_{(i-1,j+1)}$ or $L_{(i,j-1)} = L_{(i-1,j-1)}$. If both conditions hold, this is configuration (a) of criterion (ii). Next, suppose that $L_{(i,j)} = L_{(i-1,j+1)}$ and $L_{(i,j-1)} \neq$

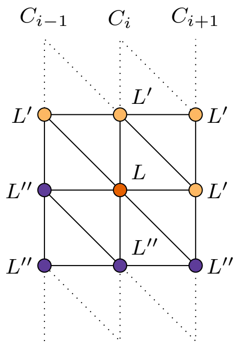


Figure 4.3: The configuration of an isolated component in Lemma 4.11(1)

$L_{(i-1,j-1)}$. If $L_{(i,j)} \neq L_{(i-1,j)}$, then we are in the scenario of Lemma 4.10(1)c, which is not possible by assumption. Hence, we must have $L_{(i,j)} = L_{(i-1,j)}$, and this is configuration (b) in criterion (ii). Similarly, if $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} = L_{(i-1,j-1)}$, then we must have $L_{(i,j-1)} = L_{(i-1,j)}$, otherwise we will be in the scenario of Lemma 4.10(1)b. Thus, we get configuration (c) of criterion (ii). We can similarly deduce criteria (iii) and (iv) from Lemma 4.10(2) and Lemma 4.10(3), respectively.

The configurations (a) to (c) in criterion (ii) are illustrated in Fig. 4.2. By Lemma 4.1, we also assume one of three analogous configurations holds for the lists on the right neighbors of (i, j) and $(i, j-1)$ under the hypothesis of criterion (ii), but for the sake of brevity we avoid listing them explicitly.

We now make the following definitions. For a list L, define the *list-class* of L in G, denoted $G[L]$, to be induced subgraph of G on those vertices v such that $L_v = L$. Let $L \in \mathcal{L}$ and let H be a (maximal connected) component of $G[L]$. If $V(H)$ is a singleton, we call H an *isolated component*, else we call H a *nonisolated component*.

Lemma 4.11. *Suppose that $\mathcal{L}$ satisfies criteria (i) to (iv).*

(1) Let H be an isolated component of a list-class $G[L]$, with $V(H) = \{(i, j)\}$. Then, there are distinct lists $L', L'' \in \mathcal{L}$ such that $L_{(i-1,j+1)} = L_{(i,j+1)} = L_{(i+1,j+1)} = L_{(i+1,j)} = L'$ and $L_{(i-1,j)} = L_{(i-1,j-1)} = L_{(i,j-1)} = L_{(i+1,j-1)} = L''$.

(2) Let H be a nonisolated component of a list-class $G[L]$, with $v \in V(H)$. Then, at least one vertical neighbor of v also belongs to $V(H)$.

Proof.

(1) Since (i, j) belongs to an isolated component of $G[L]$, the lists on its vertical neighbors are distinct from L. Let $L_{(i,j+1)} = L'$ and $L_{(i,j-1)} = L''$. By applying criterion (ii) on the vertices (i, j) and $(i, j-1)$ with respect to their left neighbors, we see that only configuration (c) can hold, else (i, j) will not belong to an isolated component. Thus, $L_{(i-1,j)} = L_{(i-1,j-1)} = L''$ and $L_{(i,j+1)} \neq L$.

Now, if $L_{(i-1,j)} \neq L'$, then we can apply criterion (ii) on the vertices $(i-1, j+1)$ and $(i-1, j)$ with respect to their right neighbors, and we see that none of the analogues of configurations (a) to (c) hold, a contradiction. Hence, $L_{(i-1,j)} = L'$.

Next, by Lemma 4.1, we also get $L_{(i+1,j+1)} = L_{(i+1,j)} = L'$ and $L_{(i,j-1)} = L''$.

Lastly, if $L' = L''$, then we will also have $L = L'$ by criterion (i), a contradiction.

(2) Since v is assumed to belong to a nonisolated component H of some list-class $G[L]$, let $u \in V(H)$ with u adjacent to v. If u lies in the same column as v, then we are done, so assume that u and v lie in distinct columns. Then, by criterion (iii), there is a vertex w adjacent to both u and v such that $w \in V(H)$. If w lies in the same column as v, then we are done, so assume that v and w lie in distinct columns. Then, u and w lie on the same column, so by criterion (iv) at least one of the vertical neighbors of v also belongs to $V(H)$.

□

The configuration in Lemma 4.11(1) is illustrated in Fig. 4.3. Note that Lemma 4.11(2) implies that for every $v \in V(H)$, where H is a nonisolated component of some list-class $G[L]$, at least one left neighbor and one right neighbor of v also belongs to $V(H)$, by criterion (i). Hence, Lemma 4.11(2) can be applied successively on vertices across columns, starting from any $v \in V(H)$. Thus, if (i, j) and $(i, j-1)$ are adjacent vertices in the column C_i with distinct lists, then using Lemma 4.11, we can pin down the possible list configurations on the nearby vertices in the columns C_{i+1} and C_{i+2} to a manageable number, as follows.

Lemma 4.12. *Suppose that $\mathcal{L}$ satisfies criteria (i) to (iv). Let $(i, j+1), (i, j) \in V(G)$ have distinct lists L_1, L_2, respectively, and suppose that neither vertex belongs to an isolated component. Then, one of the following configurations holds:*

(I) The vertices (i, k), $(i+1, k)$ and $(i+2, k)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(II) The vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(III) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(IV) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-2)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(V) The vertices (i, k), $(i+1, k)$ and $(i+2, k)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_2 for $k = j, j-1$,

and the vertex $(i+2, j)$ belongs to an isolated component of some list-class $G[L_3]$, where $L_3 \neq L_1$ and $L_3 \neq L_2$.

(VI) *The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k-1)$ and $(i+2, k-2)$ have lists identical to L_2 for $k = j, j-1$, and the vertex $(i+2, j-1)$ belongs to an isolated component of some list-class $G[L_3]$, where $L_3 \neq L_1$ and $L_3 \neq L_2$.*

(VII) *The vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_2 for $k = j, j-1$, and the vertex $(i+1, j)$ belongs to an isolated component of some list-class $G[L_3]$, where $L_3 \neq L_1$ and $L_3 \neq L_2$.*

Proof. We start with $L_{(i,j+1)} = L_1$ and $L_{(i,j)} = L_2$. By Lemma 4.11(2), this implies that $L_{(i,j+2)} = L_1$ and $L_{(i,j-1)} = L_2$. By criterion (i), $L_{(i+1,j+1)} = L_1$ and $L_{(i+1,j-1)} = L_2$. Now, again by Lemma 4.11(2), we have three cases:

(1) $L_{(i+1,j+2)} = L_1$ and $L_{(i+1,j)} = L_2$;

(2) $L_{(i+1,j)} = L_1$ and $L_{(i+1,j-2)} = L_2$;

(3) $L_{(i+1,j+2)} = L_1$, $L_{(i+1,j-2)} = L_2$ and $L_{(i+1,j)} = L_3$ where $L_1 \neq L_3$ and $L_2 \neq L_3$. In particular, by Lemma 4.11(2), $(i+1, j)$ must belong to an isolated component of the list-class $G[L_3]$.

We consider each of these cases in turn.

First, suppose case (1) holds. Then, by criterion (i), $L_{(i+2,j+1)} = L_1$ and $L_{(i+2,j-1)} = L_2$. Then, again by Lemma 4.11(2), we have three cases:

- $L_{(i+2,j+2)} = L_1$ and $L_{(i+2,j)} = L_2$. This is configuration (I).
- $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. This is configuration (II).
- $L_{(i+2,j+2)} = L_1$, $L_{(i+2,j-2)} = L_2$ and $L_{(i+2,j)} = L_3$ where $L_1 \neq L_3$ and $L_2 \neq L_3$. In particular, by Lemma 4.11(2), $(i+2, j)$ must belong to an isolated component of the list-class $G[L_3]$. This is configuration (V).

Next, suppose case (2) holds. Then, by criterion (i), $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. Again by Lemma 4.11(2), we have three cases:

- $L_{(i+2,j+1)} = L_1$ and $L_{(i+2,j-1)} = L_2$. This is configuration (III).
- $L_{(i+2,j-1)} = L_1$ and $L_{(i+2,j-3)} = L_2$. This is configuration (IV).

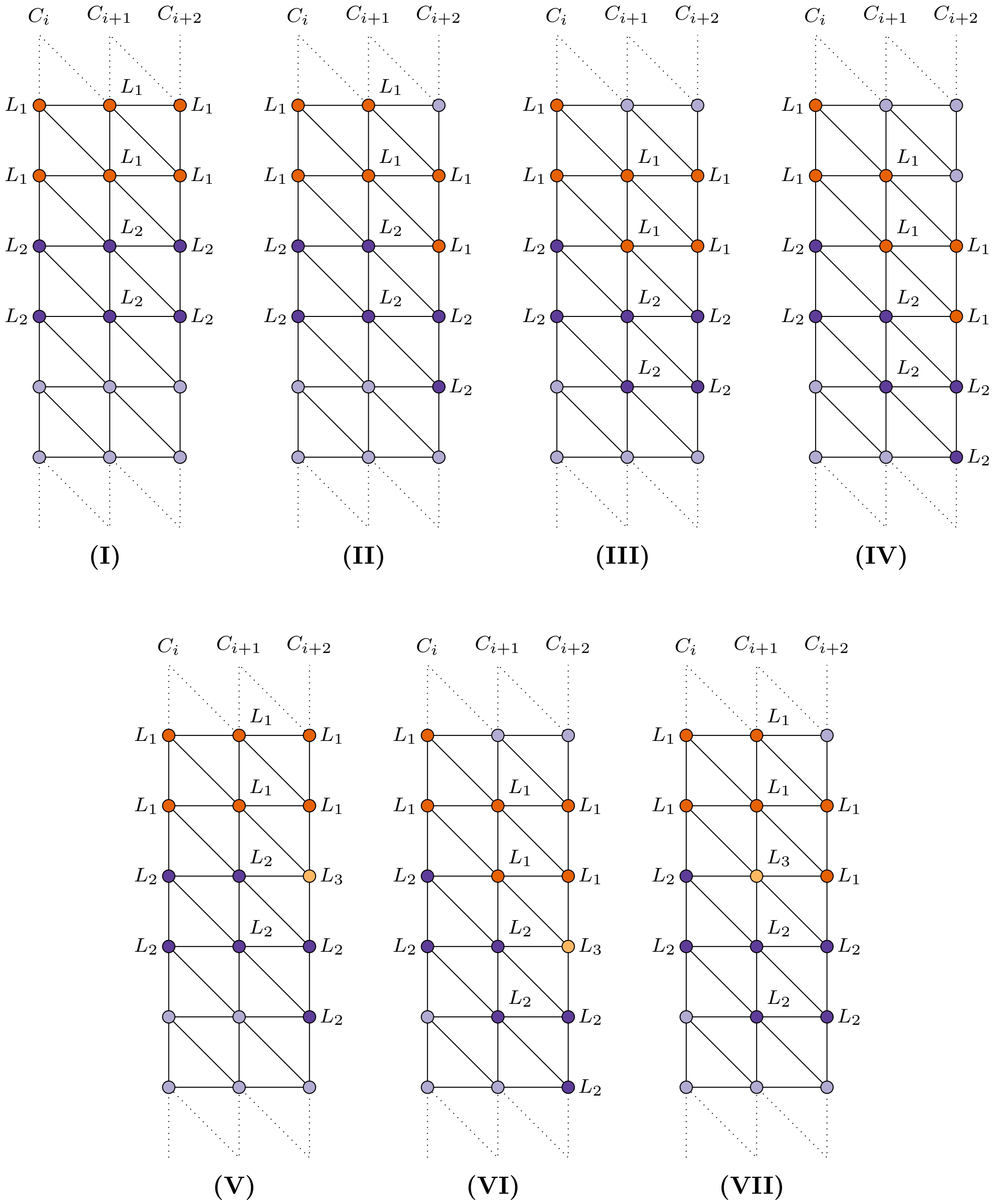


Figure 4.4: Illustration of the configurations of Lemma 4.12

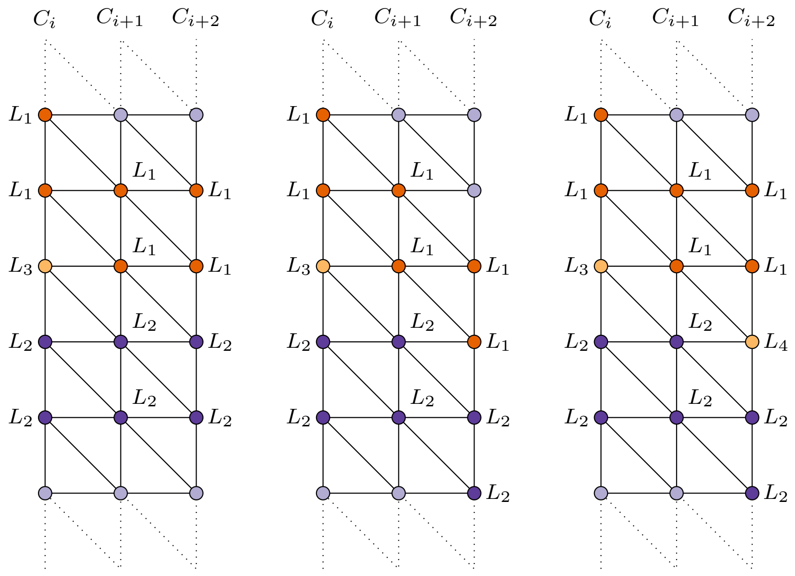


Figure 4.5: Illustration of the configurations of Lemma 4.13

- $L_{(i+2,j+1)} = L_1$, $L_{(i+2,j-3)} = L_2$ and $L_{(i+2,j-1)} = L_3$ where $L_1 \neq L_3$ and $L_2 \neq L_3$. In particular, by Lemma 4.11(2), $(i+2, j-1)$ must belong to an isolated component of the list-class $G[L_3]$. This is configuration (VI).

Lastly, suppose case (3) holds. Then, by Lemma 4.11(1), $L_{(i+2,j+1)} = L_{(i+2,j)} = L_1$ and $L_{(i+2,j-1)} = L_2$. By Lemma 4.11(2), we also have $L_{(i+2,j-2)} = L_2$. This is configuration (VII). □

Lemma 4.13. *Suppose that $\mathcal{L}$ satisfies criteria (i) to (iv). Let $(i, j+1)$, (i, j) and $(i, j-1)$ have mutually distinct lists L_1, L_3 and L_2, respectively, and suppose that (i, j) corresponds to an isolated component in $G[L_3]$. Then, one of the following configurations holds:*

(VIII) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k)$ and $(i+2, k)$ have lists identical to L_2 for $k = j-1, j-2$.

(IX) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-2)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_2 for $k = j, j-1$.

(X) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_2 for $k = j-1, j-2$, and the vertex $(i+2, j-1)$ belongs to an isolated component of some list-class $G[L_4]$, where $L_1 \neq L_4$ and $L_2 \neq L_4$, but L_4 may be identical to L_3.

Proof. We start with $L_{(i,j+1)} = L_1$, $L_{(i,j)} = L_3$ and $L_{(i,j-1)} = L_2$, with (i,j) belonging to an isolated component of the list-class $G[L_3]$. By Lemma 4.11(2), we have $L_{(i,j+2)} = L_1$ and $L_{(i,j-2)} = L_2$. By Lemma 4.11(1), we have $L_{(i+1,j+1)} = L_{(i+1,j)} = L_1$ and $L_{(i+1,j-1)} = L_2$. By Lemma 4.11(2), we also have $L_{(i+1,j-2)} = L_2$. By criterion (i), this implies that $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. Now, again by Lemma 4.11(2), we have the following three cases:

- $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. This is configuration (VIII).
- $L_{(i+2,j-1)} = L_1$ and $L_{(i+2,j-3)} = L_2$. This is configuration (IX).
- $L_{(i+2,j+1)} = L_1$, $L_{(i+2,j-3)} = L_2$ and $L_{(i+2,j-1)} = L_4$ where $L_1 \neq L_4$ and $L_2 \neq L_4$. In particular, by Lemma 4.11(2), $(i+2, j-1)$ must belong to an isolated component of the list-class $G[L_4]$. This is configuration (X). Note that L_4 may be identical to L_3.

□

These configurations are listed in Fig. 4.4 and 4.5.

We are now in a position to complete the proof of case (1) in Theorem 1.10.

4.3 Proof of case (1) in Theorem 1.10

By the results in Section 4.2, it suffices to assume that the list assignment $\mathcal{L}$ on G satisfies criteria (i) to (iv), and that, in particular, Lemma 4.13 holds. Suppose $(i, j+1)$, (i, j) and $(i, j-1)$ are three vertices in the column C_i that satisfy the hypotheses of Lemma 4.13. Then, the vertices $(i+1, j)$ and $(i+1, j-1)$ in the column C_{i+1}, as well as the vertices $(i-1, j+1)$ and $(i-1, j)$ in the column C_{i-1}, satisfy the hypotheses of Lemma 4.12. Thus, there always exists a column that has a pair of adjacent vertices that satisfies the hypotheses of Lemma 4.12, which we shall now take to be C_1 without loss of generality. Furthermore, without loss of generality, let $(1, s)$ and $(1, s-1)$ satisfy the hypotheses of Lemma 4.12.

Now, the first step of our algorithm to find an $\mathcal{L}$-coloring—which we elaborate on below—is to properly color C_1. Then, the lists on C_r all reduce to 3-lists, so C_r can be properly colored by Lemma 4.8. This in turn causes the lists on C_{r-1} to reduce to 3-lists. Thus, we can inductively color the columns from the right using Lemma 4.8 until we are only left to color the columns C_2, C_3 and C_4. Thus, it suffices to assume without loss of generality that $r = 4$.

Fix $1 \leq j \leq s$. We start with a few straightforward observations:

(A) If $L_{(1,j)} = L_{(2,j)} = L_{(2,j-1)}$, then any choice of color for $(1,j)$ will reduce the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ by 1 each. Similarly, if $L_{(1,j)} = L_{(1,j-1)} = L_{(2,j-1)}$, then any proper coloring of $(1,j)$ and $(1,j-1)$ will reduce the size of $L_{(2,j-1)}$ by 2.

(B) Consider the vertices $(1,j)$, $(2,j)$, $(2,j-1)$ and $(3,j-1)$. Suppose that a color $c \in L_{(1,j)}$ has been chosen for $(1,j)$, so the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ have potentially reduced by 1 each. Now, if $c \in L_{(3,j-1)}$ too, then coloring $(3,j-1)$ with the color c does not reduce the sizes of the residual lists on $(2,j)$ and $(2,j-1)$ any further.

(C) Suppose that $L_{(1,j)} \cap L_{(3,j-1)} = \emptyset$. If $(1,j)$ is an isolated vertex, then $(1,j+1)$, $(1,j)$, and $(1,j-1)$ satisfy the hypotheses of Lemma 4.13, and moreover these vertices must be in configuration (X). If $(1,j)$ is not an isolated vertex, then either $(1,j)$ and $(1,j-1)$ satisfy the hypotheses of Lemma 4.12, or $(1,j+1)$ and $(1,j)$ satisfy the hypotheses of Lemma 4.12, or $L_{(1,j+1)} = L_{(1,j)} = L_{(1,j-1)}$. If the first case holds, then $(1,j)$ and $(1,j-1)$ must be in configuration (I); if the second case holds, then $(1,j+1)$ and $(1,j)$ must be in configuration (IV); the third case is impossible, since by repeated application of criterion (i) we must have $L_{(1,j)} = L_{(3,j-1)}$.

Furthermore, in the case when $(1,j)$ is an isolated vertex, choosing a color for $(1,j)$ from $L_{(1,j)} \setminus L_{(2,j)}$ and for $(3,j-1)$ from $L_{(3,j-1)} \setminus L_{(2,j-1)}$ will reduce the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ by 1 each. Note that such choices are possible by criterion (i) and Lemma 4.11(1). In the other cases, any choice of color for $(1,j)$ and for $(3,j-1)$ will reduce the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ only by 1 each.

These observations are crucial for step 1 of the following two-step coloring algorithm:

1. Properly color C_1 and a set J of alternate vertices in C_3 such that the reduced list sizes on C_2 are as follows: one vertex in C_2 has a 4-list and every other vertex in C_2 has a 3-list.

2. Properly color C_4, then the remaining vertices in C_3, and finally C_2.

Assume for the moment that step 1 has been completed. Then, step 2 can be completed by repeatedly invoking Lemma 4.7 as follows.

As we shall see when we elaborate on step 1, we may assume that the 4-list in the column C_2 is on the vertex $(2,s-1)$, and that the rest of the vertices in C_2 have 3-lists. Also, the set J will turn out to be either $I := \{(3, s-2k+2) : k = 1,\dots,\lfloor s/2 \rfloor\}$ or $I' := \{(3, s-2k+1) : k = 1,\dots,\lfloor s/2 \rfloor\}$.

Now, the sizes of the lists on the remaining vertices of C_3 after the completion of step 1 are as follows: the vertices of C_3 that remain to be colored all have 3-lists; moreover, when s is odd, the vertices $(3,1)$ and $(3,2)$ each have a 4-list when the set I is colored

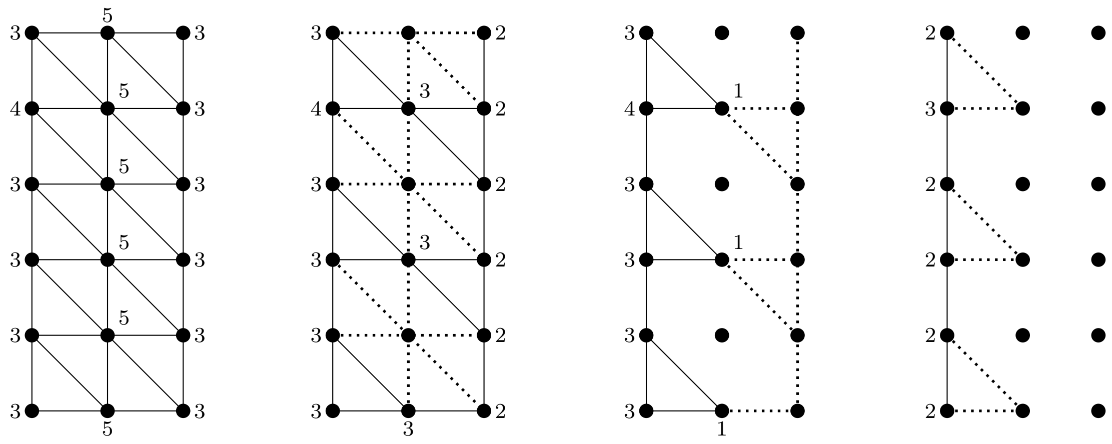


Figure 4.6: Illustration of the sizes of the lists on the columns C_2, C_3 and C_4 in step 2 when $s = 6$ and I is colored in step 1

in step 1, and the vertices $(3, 1)$ and $(3, s)$ each have a 4-list when the set I' is colored in step 1.

Next, the sizes of the lists on the column C_4 are as follows: each vertex in C_4 has a 2-list; moreover, when s is odd, the vertex $(4, 1)$ has 3-list when the set I is colored in step 1, and the vertex $(4, s)$ has a 3-list when the set I' is colored in step 1.

Thus, regardless of the parity of s, properly color the column C_4 using Lemma 4.7. This reduces the sizes of each of the remaining lists on C_3 by 2. Again by Lemma 4.7, regardless of the parity of s, properly color the remaining vertices in the column C_3. This reduces the list sizes on C_2 as follows. When s is even, each list on C_2 is reduced in size by 1. When s is odd, each list is reduced in size by 1, but for the following exception: if I is colored in step 1, then the list on $(2, 2)$ is reduced in size by 2, and if I' is colored in step 1, then the list on $(2, 1)$ is reduced in size by 2. In either case, properly color C_2 using Lemma 4.7. This completes step 2.

Fig. 4.6 and 4.7 illustrate the sizes of the lists in step 2 when s is even and odd, respectively, assuming that the set I is colored in step 1. The edges between the top and bottom rows are not shown in these figures.

We now describe step 1. If $(1, s)$ and $(1, s-1)$ are in any configuration other than (IV) and (VI), then take $J = I$, and if $(1, s)$ and $(1, s-1)$ are in configuration (IV) or (VI), then take $J = I'$, where the sets I and I' are as defined earlier in the description of step 2. From observations (A) to (C), every vertex in C_2 can have a 3-list at the end of step 1 if for every $(3, j) \in J$, either $L_{(1,j+1)} \cap L_{(3,j)} = \emptyset$, or $(1, j+1)$ and $(3, j)$ are assigned the same color. Clearly, if the lists on $(1, j+1)$ and $(3, j)$ are identical, then for any assignment of a color on $(1, j+1)$ we can pick the same color for $(3, j)$. On the other hand, if the lists

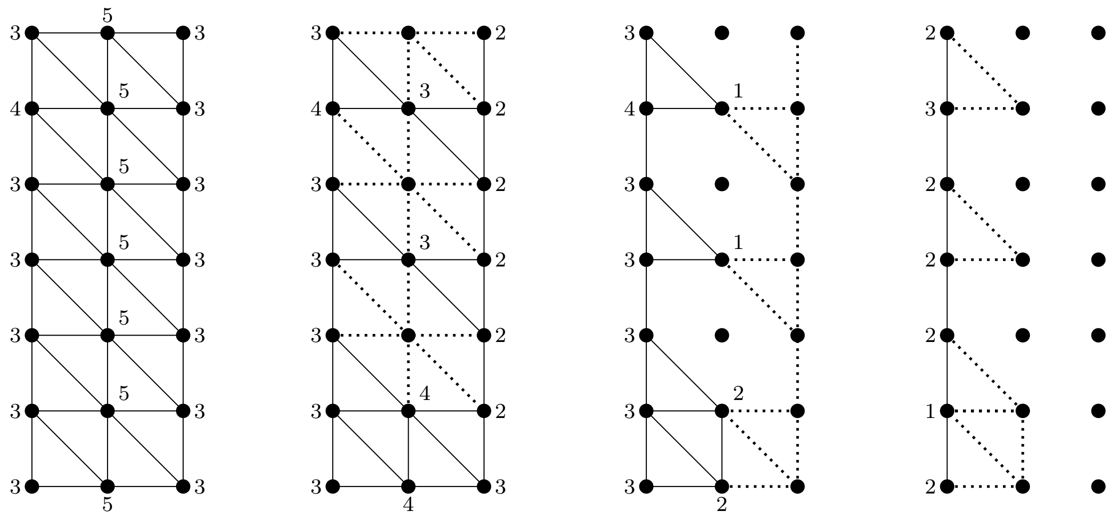


Figure 4.7: Illustration of the sizes of the lists on the columns C_2, C_3 and C_4 in step 2 when $s = 7$ and I is colored in step 1

on $(1, j+1)$ and $(3, j)$ are distinct but not disjoint, then we need to ensure that the color assigned on $(1, j+1)$ belongs to $L_{(1,j+1)} \cap L_{(3,j)}$.

So, call the pair of vertices $(1, j+1), (3, j)$ to be a *good pair* if either $L_{(1,j+1)} = L_{(3,j)}$, or $L_{(1,j+1)} \cap L_{(3,j)} = \emptyset$ and $(1, j+2)$, $(1, j+1)$, and $(1, j)$ are not in configuration (X). Define A to be the set of all pairs $(1, j+1), (3, j)$ that are not good pairs. We now carry out step 1 in three stages. In the first stage, we shall color the vertices in A. In the second stage, we color the vertices $(1, s)$ and $(1, s-1)$ in such a way that the list on $(2, s-1)$ reduces to a 4-list. Finally, we color the remaining vertices of the column C_1, followed by the remaining vertices in J.

Now, for the first stage. Suppose $(3, j) \in A$. If the lists on $(1, j+1)$ and $(3, j)$ are distinct but not disjoint, then choose a common color for $(1, j+1)$ and $(3, j)$ from $L_{(1,j+1)} \cap L_{(3,j)}$. Otherwise, we have that the lists on $(1, j+1)$ and $(3, j)$ are disjoint and the vertices $(1, j+2)$, $(1, j+1)$ and $(1, j)$ are in configuration (X). In this case, choose a color for $(1, j+1)$ from $L_{(1,j+1)} \setminus L_{(2,j+1)}$ and for $(3, j)$ from $L_{(3,j)} \setminus L_{(2,j)}$.

Note that our choice of J ensures that the vertices $(1, 1)$, $(1, s)$, $(1, s-1)$ and $(1, s-2)$ are not colored in the first stage above (cf. Fig. 4.4). So, for the second stage, pick colors for $(1, s)$ and $(1, s-1)$ as follows:

- If $(1, s)$ and $(1, s-1)$ are in any of the configurations (I) to (IV), then choose a color for $(1, s)$ from $L_{(1,s)} \setminus L_{(1,s-1)}$ and for $(1, s-1)$ from $L_{(1,s-1)} \setminus L_{(1,s)}$.
- If $(1, s)$ and $(1, s-1)$ are in configuration (V), choose a color for $(1, s)$ from $L_{(1,s)} \cap L_{(3,s-1)} \setminus L_{(1,s-1)}$ (this can be done because of criterion (i) and Lemma 4.11(1)), and

choose a color for $(1, s-1)$ from $L_{(1,s-1)} \setminus L_{(1,s)}$.

- If $(1, s)$ and $(1, s-1)$ are in configuration (VI), choose a color for $(1, s)$ from $L_{(1,s)} \setminus L_{(1,s-1)}$, and for $(1, s-1)$ from $L_{(1,s-1)} \cap L_{(3,s-2)} \setminus L_{(1,s)}$ (this can be done because of criterion (i) and Lemma 4.11(1)).
- If $(1, s)$ and $(1, s-1)$ are in configuration (VII), choose a color for $(1, s)$ from $L_{(1,s)} \setminus L_{(2,s-1)}$, and for $(1, s-1)$ from $L_{(1,s-1)} \setminus L_{(1,s)}$.

This coloring ensures that the vertex $(2, s-1)$ now has a 4-list.

Finally, for the third stage. First, color the remaining vertices in the column C_1 using Lemma 4.7. Then, color the remaining vertices in J as follows. Suppose $(3, j) \in J$ was uncolored in the first stage. If the lists on $(3, j)$ and $(1, j+1)$ assigned by $\mathcal{L}$ were identical, choose the same color on $(3, j)$ as that assigned on $(1, j+1)$. If the lists on $(3, j)$ and $(1, j+1)$ are disjoint, then choose any color for $(3, j)$ from its list.

Notice that at the end of this procedure the vertex $(2, s-1)$ still has a 4-list, and that all the other vertices in the column C_2 have 3-lists. So, this completes step 1. Combined with step 2, this completes the proof.

It is also clear from the above description that the coloring can be found in linear time.

4.4 Proofs of cases (2) and (3) in Theorem 1.10

Recall that case (2) in Theorem 1.10 says that if G is isomorphic to $T(1, s, 2)$ for $s \geq 9$, $s \neq 11$, then G is 5-choosable.

Proof of case (2) in Theorem 1.10. Let $G = T(1, s, 2)$ for $s \geq 9$, $s \neq 11$. Then, every four successive vertices $(1, j)$, $(1, j+1)$, $(1, j+2)$, $(1, j+3)$ induce a K_4. Suppose that $\mathcal{L}$ is a list assignment on G with lists of size equal to 5. Since G is 5-colorable in linear time by the results in [32, 88], it suffices to assume that not all the lists assigned by $\mathcal{L}$ are identical. Without loss of generality, suppose that $L_{(1,1)} \neq L_{(1,s)}$. Choose a color for $(1, s)$ from $L_{(1,s)} \setminus L_{(1,1)}$. Next, one can properly color the vertices $(1, s-1), (1, s-2), \ldots, (1, 7)$ in that order by successively picking a color for each vertex from its (reduced) list. Then, the lists on the remaining vertices are as follows: $(1, 6)$ has a 2-list; $(1, 1)$, $(1, 2)$ and $(1, 5)$ have 3-lists; $(1, 3)$ and $(1, 4)$ have 4-lists. There are two special cases that can be easily dealt with.

Case I: $L_{(1,2)} \cap L_{(1,6)} \neq \emptyset$.

Choose a common color for $(1,2)$ and $(1,6)$ from $L_{(1,2)} \cap L_{(1,6)}$. Then, $(1,1)$ and $(1,5)$ have 2-lists, and $(1,3)$ and $(1,4)$ have 3-lists. If we can pick a color for $(1,1)$ that does not belong to both $L_{(1,3)}$ and $L_{(1,4)}$, then we will be done by Lemma 4.7, so assume that $L_{(1,1)} \subset L_{(1,3)} \cap L_{(1,4)}$. Then, for any choice of color for $(1,1)$, the remaining 3-cycle will have 2-lists, so it will have a proper coloring only when the 2-lists are not identical, by Lemma 4.7. But, if picking $a \in L_{(1,1)}$ results in identical 2-lists being present on the remaining 3-cycle, then we instead pick the other color $a' \in L_{(1,1)} \setminus \{a\}$ for $(1,1)$ to get non-identical 2-lists on the remaining 3-cycle.

So, it suffices to assume that $L_{(1,2)} \cap L_{(1,6)} = \emptyset$. Denote by K_4^- the complete graph on four vertices with an edge removed.

Case II: $L_{(1,1)} \cap L_{(1,5)} \neq \emptyset$.

Choose a common color c for $(1,1)$ and $(1,5)$ from $L_{(1,1)} \cap L_{(1,5)}$. If $c \notin L_{(1,2)}$, then the remaining vertices form a K_4^- with three 3-lists and one 1-list, and one can see that a proper coloring can always be found from this configuration of lists. If $c \in L_{(1,2)}$, then we have a K_4^- with three 2-lists and one 3-list. Since $L_{(1,2)} \cap L_{(1,6)} = \emptyset$ by assumption, we can choose a color for either $(1,2)$ or $(1,6)$ that does not belong to $L_{(1,3)}$. Then, we can properly color the rest of the vertices using Lemma 4.7.

So, we additionally assume that $L_{(1,1)} \cap L_{(1,5)} = \emptyset$.

Now, choose a color for $(1,3)$ from $L_{(1,3)} \setminus L_{(1,6)}$. Then, the lists are now as follows: $(1,4)$ has a 3-list; $(1,2)$ and $(1,6)$ have 2-lists; lastly, either $(1,1)$ has a 2-list and $(1,5)$ has a 3-list, or vice-versa, since the color chosen for $(1,3)$ can belong to at most one of $L_{(1,1)}$ and $L_{(1,5)}$. In either case, a color can be chosen for $(1,4)$ such that both $(1,1)$ and $(1,5)$ end up with 2-lists. The lists on $(1,2)$ and $(1,6)$ are now a 1-list and a 2-list, not necessarily in that order, since the color chosen for $(1,4)$ can belong to at most one of $L_{(1,2)}$ and $L_{(1,6)}$. In any case, the remaining four vertices form a path graph with three 2-lists and one 1-list, so a proper coloring can be found using Lemma 4.7.

This completes the proof. It is also clear that the list coloring can be found in linear time. □

Recall that case (3) in Theorem 1.10 says that if G is a simple graph isomorphic to $T(2,s,t)$ where s and t are both even, then G is 5-choosable. The following lemma will be repeatedly invoked in the proof of this case.

Lemma 4.14. *Let $G = K_4^-$ be the complete graph on four vertices with an edge removed, where $V(G) = \{a,b,x,y\}$ and $\{x,y\}$ is an independent set. Suppose that $\mathcal{L}$ is a list* *assignment on G such that no list is empty and $|L_a| + |L_b| = |L_x| + |L_y|$. Then, one can choose colors for x and y such that the sizes of the lists on a and b reduce by 1 each.*

Proof. If $L_x \cap L_y \neq \emptyset$, then choose a common color for x and y from $L_x \cap L_y$. Clearly, this reduces the sizes of the lists on a and b by 1 each.

So, suppose $L_x \cap L_y = \emptyset$. If $L_x \not\subset L_a \cup L_b$, then we can choose a color for x from $L_x \setminus (L_a \cup L_b)$, and any color for y from L_y, to reduce the sizes of the lists on a and b by 1 each. Similarly, when $L_y \not\subset L_a \cup L_b$ we are done.

So, suppose that $L_x, L_y \subset L_a \cup L_b$. But then there are k distinct available colors for x and y together, where $k = |L_x| + |L_y|$, as well as for a and b together. Thus, any color in L_x can belong to at most one of L_a and L_b, and similarly for any color in L_y. Moreover, it is not possible that all of the k available colors on x and y belong to a single list (say, L_a), because the other list (i.e., L_b) will then be empty, which is not allowed by assumption. Therefore, it is possible to choose colors for x and y from their respective lists such that each color belongs to a different list between L_a and L_b. This reduces the sizes of the lists on a and b by 1 each. □

Proof of case (3) in Theorem 1.10. Let $G = T(2, s, t)$ for even $s \geq 4$ and even $t \neq 0$, $s - 2$. By the remarks following Theorem 1.1, it follows that the graphs $T(2, s, t)$ and $T(2, s, s-t-2)$ are isomorphic, so without loss of generality we assume that $2 \leq t \leq \frac{s}{2} - 1$.

Our strategy is to properly color the column C_2 in such a way that the lists on the column C_1 are all reduced to 2-lists, so that C_1 can then be properly colored using Lemma 4.7. Note that, for every j, the vertices $(1, j)$, $(1, j-1)$, $(2, j-1)$ and $(2, j+t)$ form a K_4^- with the vertices on the column C_2 forming an independent set. This suggests the following scheme of coloring.

Fix $1 \leq j \leq s$. Using Lemma 4.14, we can color $(2, j-1)$ and $(2, j+t)$ such that the lists on $(1, j)$ and $(1, j-1)$ reduce in size by 1 each. Then, regardless of how the rest of the neighbors of $(1, j)$ and of $(1, j-1)$ in the column C_2 are colored, the end result is that the lists on these two vertices reduce to 2-lists, as required. If we can do this for each even j, then we will have colored C_2 in such a way that the lists on C_1 all reduce to 2-lists. A little bit of care is required to ensure that this can be done for every even j, while also ensuring that the coloring on C_2 is proper. The details now follow.

Suppose that $t < \frac{s}{2} - 1$.

- For each $j \in \{2, 4, \ldots, t\}$, we may use Lemma 4.14 to color the vertices of the form $(2, j-1)$ and $(2, j+t)$.

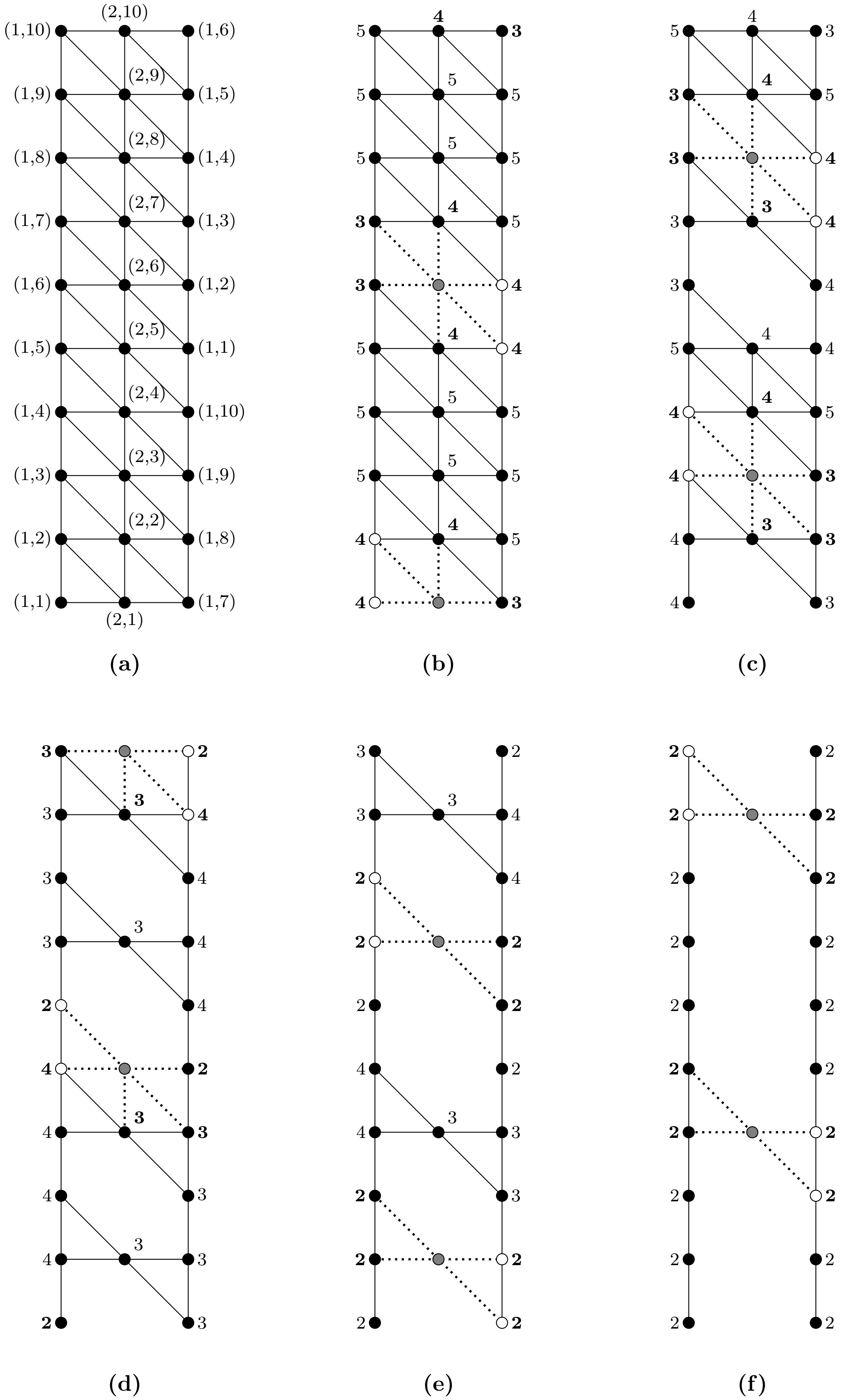


Figure 4.8: Illustration of the proof of case (3) in Theorem 1.10 for $G = T(2, 10, 4)$

- For $j = t + 2$, we need to color $(2, t + 1)$ and $(2, 2t + 2)$, but notice that the list on $(2, t + 1)$ has been reduced to a 4-list, since $(2, t + 2)$ has already been colored (when $j = 2$). However, the list on $(1, t + 2)$ has also been reduced to a 4-list, so Lemma 4.14 is still applicable.

- For each $j \in \{t + 4, t + 6, \ldots, s - t - 2\}$, notice that the list on $(2, j - 1)$ has been reduced to a 3-list, but the lists on $(1, j)$ and $(1, j - 1)$ have also been reduced to 4-lists each, so Lemma 4.14 is still applicable.

- For $j = s - t$, notice that the list on $(2, s - t - 1)$ is reduced to a 3-list and the list on $(2, s)$ is reduced to a 4-list, but the list on $(1, s - t)$ is reduced to a 3-list and the list on $(1, s - t - 1)$ is reduced to a 4-list. So, Lemma 4.14 is still applicable.

- Lastly, for each $j \in \{s - t + 2, s - t + 4, \ldots, s\}$, notice that the lists on $(2, j - 1)$ and $(2, j + t)$ are reduced to 3-lists each, but the lists on $(1, j)$ and $(1, j - 1)$ have also been reduced to 3-lists each, so Lemma 4.14 is still applicable.

Next, consider the case when $t = \frac{s}{2} - 1$. Since $t + 2 = s - t$, some of the cases above reduce to a single degenerate case, as explained below:

- As before, for each $j \in \left\{2, 4, \ldots, \frac{s}{2} - 1\right\}$, we may use Lemma 4.14 to color the vertices of the form $(2, j - 1)$ and $(2, j + t)$.

- For $j = \frac{s}{2} + 1$, we need to color $\left(2, \frac{s}{2}\right)$ and $(2, s)$, but notice that the lists on $\left(2, \frac{s}{2}\right)$ and $(2, s)$ have been reduced to 4-lists. However, the list on $\left(1, \frac{s}{2} + 1\right)$ has also been reduced to a 3-list, so Lemma 4.14 is still applicable.

- Lastly, just as before, for each $j \in \left\{\frac{s}{2} + 3, \frac{s}{2} + 5, \ldots, s\right\}$, notice that the lists on $(2, j - 1)$ and $(2, j + t)$ are reduced to 3-lists each, but the lists on $(1, j)$ and $(1, j - 1)$ have also been reduced to 3-lists each, so Lemma 4.14 is still applicable.

Thus, the coloring scheme outlined in the beginning can be implemented over all even j to get a proper coloring of C_2 so that the lists on C_1 are all reduced to 2-lists. The proof is completed by using Lemma 4.7 to properly color C_1. Clearly, the list coloring is found in linear time. □

Fig. 4.8 illustrates the coloring algorithm for $G = T(2, 10, 4)$.

4.5 Proof of case (4) in Theorem 1.10

Recall that case (4) in Theorem 1.10 says that if the simple graph $G = T(r, s, t)$ is 3-chromatic, then it is 5-choosable.

Lemma 4.15. *Let G be a 3-regular bipartite graph. Let $V(G)$ be partitioned into two independent sets A and B. Let $\mathcal{L}$ be a list assignment that assigns a list of size 3 to each vertex in A and a list of size 2 to each vertex in B. Then, G is $\mathcal{L}$-choosable. Moreover, such a list coloring can be found in linear time.*

Proof. Find a perfect matching M in G and consider the graph $G - M$. Since $G - M$ is a 2-regular bipartite graph, it breaks up into a union of disjoint even cycles. Place an orientation on the edges of G such that each of these cycles becomes a directed cycle, and such that the edges in M are oriented from A to B. Then, $\mathsf{outdegree}(v) = 2$ for every $v \in A$ and $\mathsf{outdegree}(w) = 1$ for every $w \in B$. Note that there are no odd directed cycles in this orientation, since G is bipartite. Thus, by Lemma 4.6, G is $\mathcal{L}$-choosable for any list assignment that assigns a list of size 3 to each vertex in A and a list of size 2 to each vertex in B.

Furthermore, Theorem 4.3 shows that M can be found in $O(|E|)$ time, which is also $O(|V|)$ time, since G is of bounded degree. Thus, the list coloring can be found in linear time. □

In Chapter 3, we proved the following result:

Theorem 3.10. *Let $G = T(r, s, t)$ be a simple 6-regular triangulation of the torus. If G is 3-chromatic, then G is 5-choosable.*

The proof of this theorem is entirely algorithmic, except for one use of the theorem of Alon–Tarsi (Theorem 3.6) to show that a toroidal 3-regular bipartite graph is $\mathcal{L}$-choosable for a list assignment $\mathcal{L}$ as in the hypothesis of Lemma 4.15. Using the proof of Lemma 4.15 in its place, we obtain the proof of case (4) in Theorem 1.10 as a corollary:

Corollary 4.16. *Every simple 3-chromatic 6-regular toroidal triangulation is 5-choosable. Moreover, a 5-list coloring can be found in linear time.*

4.6 Completing the proof of Theorem 1.10

Lastly, we show that all the simple 3-chromatic graphs $T(r, s, t)$ are not 3-choosable. Note that $T(r, s, t)$ is 3-chromatic if and only if $s \equiv 0 \equiv r - t \pmod 3$. Let

$$L_1 := \{1, 2, 3\}, \quad L_2 := \{2, 3, 4\}, \quad L_3 := \{1, 3, 4\}.$$

4.6.1 The graphs $T(r, s, t)$ for $r \geq 4$, $s \geq 3$

Let $\mathcal{L}$ be the list-assignment that assigns the above lists to the columns of $T(r, s, t)$ as follows:

$$L_1 : C_1, C_2;$$
$$L_2 : C_3;$$
$$L_3 : C_4, \ldots, C_r.$$

Let the vertices $(1, 1)$ and $(1, 2)$ be properly colored using $\mathcal{L}$ in any manner. This uniquely determines a proper coloring of the induced subgraph on $C_1 \cup C_2$.

Now, there is a unique way to extend this coloring properly to the induced subgraph on $C_2 \cup C_3$ as follows: simply extend the coloring from C_2 to C_3 using the same lists used on C_2, namely $L_1 = \{1, 2, 3\}$; then, recolor all the vertices in C_3 that have the color 1 with the color 4. The reason behind this is as follows:

- whenever there exist two adjacent vertices in C_2 that are colored using $\{2, 3\} \subset L_1 \cap L_2$, the common neighbor of these two vertices in C_3 must receive the color 4;
- in any proper coloring of $C_1 \cup C_2$, there will be $s/3$ pairs of vertices in C_2 that are colored using $\{2, 3\}$, and no two of these pairs are adjacent in C_2;
- if a vertex in C_3 has its color fixed to be 4 as above, then the colors of its two vertical neighbors are also fixed.

In this manner, one can see that the coloring is extended uniquely to the rest of C_3, with 4 occuring in those places where 1 would have occured had C_3 also been colored using $L_1 = \{1, 2, 3\}$.

Next, repeat the same process to extend the coloring on C_3 to a proper coloring on the induced subgraph on $C_3 \cup C_4 \cup \cdots \cup C_r$ as follows: color the vertices in $C_4 \cup \cdots \cup C_r$ using the colors used on C_3, namely $L_2 = \{2, 3, 4\}$, and then recolor those vertices in $C_4 \cup \cdots \cup C_r$ that have the color 2 with the color 1.

Now, we note that this coloring cannot be proper on all of $T(r, s, t)$ because this process of successive relabelling has mapped the tuple $(1, 2, 3)$ to $(2, 1, 3)$. Thus, for this to be a proper coloring of $T(r, s, t)$, the original coloring on C_1 must arise as the unique extension of the coloring on C_r to the induced subgraph on $C_r \cup C_1$; but, $(2, 1, 3)$ is not a cyclic permutation of $(1, 2, 3)$, so this cannot happen for any t.

4.6.2 The graphs $T(2, s, t)$ for $s \geq 6$

First, consider the case $s \geq 12$. Since $T(2, s, t)$ is assumed to be simple, we ignore the case $t = s - 1$. Next, by the remarks in Section 4.1, $T(2, s, t)$ is isomorphic to $T(2, s, s - t - 2)$. Furthermore, $t \equiv 2 \pmod 3$ since $T(2, s, t)$ is assumed to be 3-chromatic. Hence, it suffices to assume that either $t = s - 4$, or t lies in the range $5 \leq t \leq \lfloor s/2 \rfloor - 1$.

Now, let $R_1, \ldots, R_s$ denote the s rows of $T(2, s, t)$. Let $\mathcal{L}$ be the list assignment that assigns the lists L_1, L_2, L_3 to the rows of $T(2, s, t)$ as follows, where $s = 3\ell$:

$$\begin{aligned} L_1 &: R_1, \ldots, R_\ell; \\ L_2 &: R_{\ell+1}, \ldots, R_{2\ell}; \\ L_3 &: R_{2\ell+1}, \ldots, R_s. \end{aligned}$$

Fig. 4.9 illustrates this list assignment for the graph $T(2, 12, 5)$; the vertices colored red, yellow, and blue are assigned the lists L_1, L_2, and L_3, respectively.

Let the vertices $(1, 1)$ and $(1, 2)$ be properly colored using $\mathcal{L}$ in any manner. This uniquely determines a proper coloring of the first ℓ rows. Now, we can find a vertex v in one of the remaining blocks of size ℓ such that the residual list on v has size equal to 1, as follows:

- If the colors on $(1, \ell)$ and $(2, \ell)$ are both present in L_2, then take $v = (1, \ell + 1)$.
- If the color on $(1, \ell)$ is 1, then the pairs $(1, \ell - 1), (1, \ell - 2)$ and $(2, \ell), (2, \ell - 1)$ are colored using $\{2, 3\}$. Additionally, if the color on $(2, \ell)$ is 2, then the pair $(2, \ell - 1), (2, \ell - 2)$ is colored using $\{1, 3\}$, and if the color on $(2, \ell)$ is 3, then the pair $(1, \ell), (1, \ell - 1)$ is colored using $\{1, 3\}$.

  From this data, one choice for v is given by:

$$v = \begin{cases} (2, \ell - 1 + t), & 5 \leq t \leq \ell + 1; \\ (1, \ell - 2 - t), & \ell + 2 \leq t \leq \lfloor s/2 \rfloor - 1 \text{ and } (2, \ell) \text{ is colored } 2; \\ (2, \ell + t), & \ell + 2 \leq t \leq \lfloor s/2 \rfloor - 1 \text{ and } (2, \ell) \text{ is colored } 3; \\ (1, \ell + 3), & t = s - 4. \end{cases}$$

- A similar analysis can be done when $(2, \ell)$ is given the color 1. In that case, one

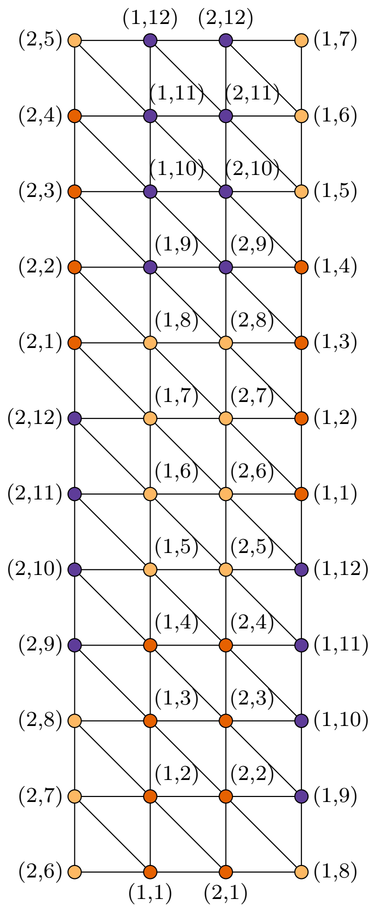


Figure 4.9: Illustration of non-$\mathcal{L}$-colorable 3-list-assignments $\mathcal{L}$ on 3-chromatic graphs $T(2,s,t)$ for $s \geq 12$ through $G = T(2,12,5)$. Distinct colors denote distinct lists among L_0, L_1, L_2.

choice for v is:

$$v = \begin{cases} (2, \ell - 2 + t), & 5 \leq t \leq \ell + 1; \\ (1, \ell - 1 - t), & \ell + 2 \leq t \leq \lfloor s/2 \rfloor - 1 \text{ and } (1, \ell) \text{ is colored } 2; \\ (2, \ell + t), & \ell + 2 \leq t \leq \lfloor s/2 \rfloor - 1 \text{ and } (1, \ell) \text{ is colored } 3; \\ (1, \ell + 2), & t = s - 4. \end{cases}$$

This allows one to extend the coloring to the entire block of ℓ vertices in which v belongs. Finally, we can repeat the process to color the third block of ℓ vertices. But, this is not a proper coloring of $T(2,s,t)$ for the same reason as in the previous case.

The remaining cases when $r = 2$ are $T(2,6,2)$, $T(2,9,2)$ and $T(2,9,5)$. Since the last two are isomorphic to each other, we shall only consider $T(2,9,5)$. In this case, we use the same list assignment as in the case when $s \geq 12$, and we note that after the first block of ℓ vertices is colored we can choose v to be either $(1,6)$ or $(1,5)$, depending on whether the color 1 is given to $(1,3)$ or $(2,3)$, respectively. The only graph left to consider is $T(2,6,2)$, which we handle in an ad hoc manner in Section 4.8.

4.6.3 The graphs $T(3, s, t)$ for $s \geq 3$

First, consider the case $s \geq 12$ and $t \neq 0$. By the remarks in Section 4.1, $T(3, s, t)$ is isomorphic to $T(3, s, s - t - 3)$, and $t \equiv 0 \pmod 3$ since $T(3, s, t)$ is assumed to be 3-chromatic. Hence, it suffices to assume that t lies in the range $3 \leq t \leq \lfloor (s-3)/2 \rfloor$. We use the same list assignment on these graphs as in the case $T(2, s, t)$ for $s \geq 12$. A similar analysis shows that $T(3, s, t)$ is not 3-choosable in these cases, so we omit the details.

The only remaining cases are $T(3, s, 0)$ for $s \geq 3$, $T(3, 6, 3)$, $T(3, 9, 3)$ and $T(3, 9, 6)$. It is easy to see that the same list assignment as above also works for $T(3, s, 0)$ for $s \geq 6$. Also, the graph $T(3, 3, 0)$ is isomorphic to $K_{3,3,3}$, which is known to be 4-list chromatic [59]. Lastly, the graphs $T(3, 6, 0)$ and $T(3, 6, 3)$ are isomorphic to each other, and so are $T(3, 9, 0)$ and $T(3, 9, 6)$. So, the only graph left to consider is $T(3, 9, 3)$, which we handle in an ad hoc manner in Section 4.8.

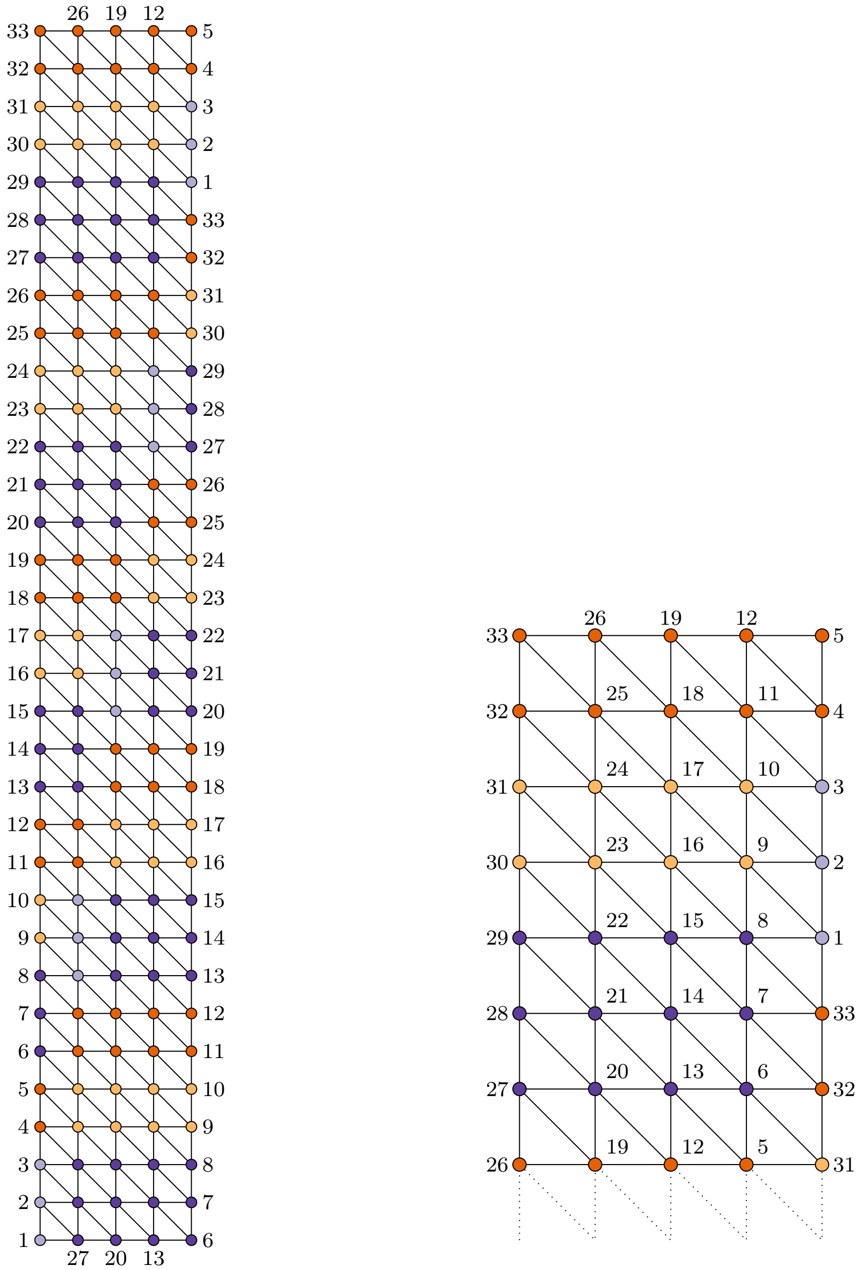
4.6.4 The graphs $T(1, s, t)$ for $s \geq 9$

Suppose that the 3-chromatic graph $T(1, s, t)$ is not isomorphic to $T(r', s', t')$ for any $r' > 1$. This happens if and only if $\gcd(s, t) = 1 = \gcd(s, t + 1)$, so we must have $s \equiv 3 \pmod 6$. By the remarks in Section 4.1, $T(1, s, t)$ is isomorphic to $T(1, s, s - t - 1)$, so we can assume $0 \leq t \leq \lfloor (s-1)/2 \rfloor$. Moreover, $T(1, s, t)$ has loops when $t = 0$ and has multiple edges when $t = 1, \lfloor (s-1)/2 \rfloor$, so we ignore these cases. Since we assume that $T(1, s, t)$ is 3-chromatic, we also have $t \equiv 1 \pmod 3$. Thus, it suffices to consider only those t in the range $4 \leq t \leq (s-7)/2$. Note that the least value of s for which there exists some t in the above range and for which $\gcd(s, t) = 1 = \gcd(s, t + 1)$ is $s = 21$.

For simplicity, we label the vertex $(1, j)$ with the integer j (recall that j is taken modulo s). We shall use the following modifications of the above coloring scheme.

First, suppose that $7 \leq t < (s + 1)/4$. Fix L_0 to be an arbitrary 3-list. Let $\mathcal{L}$ be the list assignment on $T(1, s, t)$ that assigns the lists L_0, L_1, L_2, L_3 as follows:

$$\begin{aligned} &L_1 : \{s - kt, s - 1 - kt : k = 0, 1, 2, 3, 4\}, \\ &L_2 : \{s - 2 - kt, s - 3 - kt : k = 0, 1, 2, 3\}, \\ &L_3 : \{s - 4 - kt, \ldots, s - t + 1 - kt : k = 0, 1, 2, 3\}, \end{aligned}$$



(a) $T(1, 33, 7)$

(b) Close-up of $T(1, 33, 7)$

Figure 4.10: Illustration of non-$\mathcal{L}$-colorable 3-list-assignments $\mathcal{L}$ on 3-chromatic graphs $T(1, s, t)$ for $s \geq 9$ and $7 \leq t < (s+1)/4$ through $G = T(1, 33, 7)$. Distinct colors denote distinct lists among L_0, L_1, L_2, L_3.

and any remaining vertices are assigned the list L_0. Fig. 4.10a illustrates this list assignment for the graph $T(1, 33, 7)$; the vertices colored lilac, red, yellow, and blue are assigned the lists L_0, L_1, L_2, and L_3, respectively.

Essentially the same arguments as before work in this case as well, so we omit the details from here onward. Fig. 4.10b shows a selected portion of Fig. 4.10a on which a similar argument as in the previous cases can be applied to show that $T(1, 33, 7)$ is not 3-choosable.

Next, suppose that $(s + 6)/4 \leq t < (s - 3)/3$. Define $\mathcal{L}$ as follows:

$$\begin{aligned}
L_1 &: \{s - kt, s - 1 - kt : k = 0, 1, 2, 3, 4\},\\
L_2 &: \{s - 2 - kt, s - 3 - kt : k = 0, 1, 2, 3, 4\},\\
L_3 &: \{s - 4 - kt, s - 5 - kt : k = 0, 1, 2, 3, 4\}\\
&\quad \cup \{s - 6 - kt, \ldots, 2s - 4t + 1 - kt : k = 0, 1, 2\}\\
&\quad \cup \{2s - 4t - kt, \ldots, 2s - 4t - 5 - kt : k = 1, 2\}\\
&\quad \cup \{2s - 4t - 6 - kt, \ldots, s - t + 1 - kt : k = 0, 1, 2\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0.

Next, suppose that $(s + 3)/3 < t \leq (s - 7)/2$. Define $\mathcal{L}$ as follows:

$$\begin{aligned}
L_1 &: \{s - kt, s - 1 - kt : k = 0, 1, 2, 3, 4\},\\
L_2 &: \{s - 2 - kt, s - 3 - kt : k = 0, 1, 2, 3, 4\},\\
L_3 &: \{s - 4 - kt, s - 5 - kt : k = 0, 1, 2, 3, 4\}\\
&\quad \cup \{s - 6 - kt, \ldots, 2s - 3t + 1 - kt : k = 0, 1\}\\
&\quad \cup \{2s - 3t - 2 - kt, \ldots, s - t + 1 - kt : k = 0, 1\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0.

The remaining cases are when $t = 4, (s + 1)/4, (s - 3)/3, (s + 3)/3$. These are handled by modifying these list assignments appropriately, as we show in Section 4.8.

The proofs of Theorem 1.10 and Corollary 1.11 are now complete from the above results in Sections 4.3 to 4.6.

4.7 Concluding remarks and further questions

In the remarks following the statement of Theorem 1.10 in Section 1.3.2, we noted that Theorem 1.10 excludes only a small, finite set of 5-chromatic graphs, as well as an infinite subset of 4-chromatic graphs, both of the form $T(1, s, t)$. We shall elaborate on these details now.

4.7.1 The simple graphs $T(r, s, t)$ for $r < 4$ or $s < 3$

We first look at the graphs that are not covered by case (1) in Theorem 1.10; these are the graphs $T(r, s, t)$ with $r < 4$ or $s < 3$. But, in particular, we need only be concerned with the choosability of the simple graphs among these, because if $T(r, s, t)$ is a loopless multigraph, then we may remove the duplicated edges to get a simple d-regular graph for $d \leq 5$, which is 5-choosable by Brooks's theorem (Theorem 3.4), except when the graph is isomorphic to K_6 (but this happens only when $(r, s) \in \{(1, 6), (2, 3), (3, 2)\}$). Moreover, these graphs can be 5-list colored in linear time [89].

Now, one can check that the graphs $T(r, s, t)$ with $s < 3$ either contain loops or multiple edges, so it suffices to assume $r < 4$ and $s \geq 3$.

For $r = 3, s \geq 3$, there are no graphs $T(3, s, t)$ with loops or multiple edges.

For $r = 2, s \geq 3$, there are no graphs with loops, and the loopless multigraphs are precisely those with $t = 0,\ s - 2,\ s - 1$, so we assume that $1 \leq t \leq s - 3$. In particular, it suffices to assume that $s \geq 4$ in this case.

For $r = 1, s \geq 3$, the graph $T(1, s, t)$ is isomorphic to $T(1, s, s - t - 1)$, so it suffices to consider the values of t in the range $0 \leq t \leq \lfloor (s - 1)/2 \rfloor$. Then, the graphs $T(1, s, t)$ with loops are precisely those with $t = 0$, and the loopless multigraphs are precisely those with $t = 1,\ \lfloor (s - 1)/2 \rfloor$. So, when $r = 1$, we need only consider the graphs $T(1, s, t)$ with $2 \leq t \leq \lfloor (s - 1)/2 \rfloor - 1$. In particular, it suffices to assume that $s \geq 7$ in this case.

The graphs $T(3, s, t)$ for $s \geq 3$

The normal circuits for $T(3, s, t)$ have lengths s, $3s/\gcd(s, t)$ and $3s/\gcd(s, t + 3)$. Again, if either $3s/\gcd(s, t)$ or $3s/\gcd(s, t + 3)$ is at least 4, then $T(3, s, t)$ is isomorphic to $T(r', s', t')$ for some $r' \geq 4$, and we are done by case (1) in Theorem 1.10. So, assume that both are at most 3. If either one equals 3, then so does the other, and $T(3, s, t)$ is in fact 3-chromatic in this case, so it is 5-choosable by case (4) in Theorem 1.10. So, assume that both are at most 2. But, it is not possible that both $\gcd(s, t)$ and $\gcd(s, t + 3)$ equal 2. Note that all the above 5-list colorings can be found in linear time as well. The only case left is when $T(3, s, t)$ is isomorphic to $T(1, 3s, t'')$ and it does not satisfy any of the cases (1) to (4).

The graphs $T(2, s, t)$ for $s \geq 4$

The normal circuits of $T(2, s, t)$ have lengths s, $2s/\gcd(s, t)$ and $2s/\gcd(s, t+2)$. If either $2s/\gcd(s, t)$ or $2s/\gcd(s, t+2)$ is at least 4, then $T(2, s, t)$ is isomorphic to $T(r', s', t')$ for some $r' \geq 4$, and so it is 5-choosable by case (1) in Theorem 1.10. So, suppose that both $2s/\gcd(s, t)$ and $2s/\gcd(s, t+2)$ are at most 3. Note that both cannot be equal to 3 simultaneously. If any one equals 2, then so does the other, and this case is covered by case (3) in Theorem 1.10. All the above 5-list colorings can clearly be found in linear time as well. The only case left is when $T(2, s, t)$ is isomorphic to $T(1, 2s, t'')$ and it does not satisfy any of the cases (1) to (4).

The graphs $T(1, s, t)$ for $s \geq 7$

When $s = 7$, we have to only consider the case $t = 2$, and $T(1, 7, 2)$ is isomorphic to K_7, which is both 7-chromatic and 7-list chromatic. When $s = 11$, we have to consider the cases $t = 2$, 3, 4, but in each case the graph is isomorphic to the 6-chromatic triangulation J of Albertson and Hutchinson [2]. By Dirac's map color theorem for choosability [22], the graph J is also 6-list chromatic. So, assume that $s \geq 8$ and $s \neq 11$.

As shown by Yeh and Zhu [104], other than a small, finite list of exceptions, the simple 5-chromatic 6-regular toroidal triangulations are those isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$, $s \geq 9$, $s \neq 11$. Case (2) of Theorem 1.10 shows that the graphs $T(1, s, 2)$ for $s \geq 9$, $s \neq 11$ are all 5-choosable in linear time. Thus, we obtain an infinite class of 5-chromatic-choosable simple toroidal triangulations, proving Corollary 1.11.

Thus, the only graphs that are not covered by Theorem 1.10 are of the form $T(1, s, t)$. Moreover, these consist only of the finitely many 5-chromatic graphs not of the form $T(1, s, 2)$, as well as the 4-chromatic graphs not covered by cases (1) to (3). We are presently unable to comment on the choosability of these remaining graphs.

4.7.2 Further questions

As shown by Yeh and Zhu [104], there is a small, finite set of 5-chromatic graphs of the form $T(1, s, t)$ that are not isomorphic to $T(1, s, 2)$, with the largest among them having 33 vertices. This suggests the following natural question:

Question 4.17. *Is $\chi_\ell(G) = 5$ for the finitely many 6-regular toroidal triangulations that are 5-chromatic and not isomorphic to $T(1, s, 2)$ for any s?*

In Chapter 3, we asked whether any of the 3-chromatic 6-regular toroidal triangulations are 5-list chromatic (Question 3.15). In light of the results in this chapter, we consider a similar question for the 4-chromatic triangulations not covered in Theorem 1.10.

Question 4.18. *Is $\chi_\ell(G) \in \{4,5\}$ for every 4-chromatic 6-regular toroidal triangulation? That is, does there exist a 4-chromatic 6-list chromatic graph $T(r,s,t)$?*

In Chapter 1, we defined the *jump* of a graph G, $\mathsf{jump}(G)$, to be $\chi_\ell(G) - \chi(G)$ (Definition 1.2). In Chapter 3, we showed that every loopless 6-regular toroidal triangulation satisfies $\mathsf{jump}(G) \leq 2$. The largest jump for any toroidal graph (which we defined as $\mathsf{jump}(g)$ for $g = 0$) is at least 2 since there exist 3-chromatic planar graphs that are 5-list chromatic [70, 101]. However, we do not have any "legitimate" example of a nonplanar toroidal graph G that satisfies $\mathsf{jump}(G) = 2$.

Question 4.19. *Does there exist a nonplanar toroidal graph G with $\mathsf{jump}(G) \geq 2$ and such that any planar subgraph H of G has $\mathsf{jump}(H) < 2$?*

We note that such "legitimate" examples must exist as the genus g increases: we have shown [18] that for connected graphs embeddable on an orientable surface with genus $g > 0$, the largest jump among the r-chromatic graphs is of the order $o(\sqrt{g})$ when r is of the order $o(\sqrt{g}/\log_2(g))$, so graphs with small chromatic number (in particular, any planar graph) cannot be the sole examples of graphs attaining a large jump on a surface of large genus $g > 0$.

4.8 Miscellaneous cases in Theorem 1.10

We describe 3-list assignments on the 3-chromatic graphs not discussed in Section 4.5, in order to show that they are not 3-choosable. Similar arguments as described in Section 4.6.4 will show that these graphs are not 3-choosable for the given list assignments, so we omit the details.

First, consider $T(1,s,4)$ for $s \geq 21$. Let $\mathcal{L}$ be the list assignment that assigns the lists L_0, L_1, L_2, L_3 as follows:

$$\begin{aligned} L_1 &: \{s-kt, s-1-kt : k = 0,1,2,3,4\}, \\ L_2 &: \{s-2-kt : k = 0,1,2,3\}, \\ L_3 &: \{s-3-kt : k = 0,1,2,3\}, \end{aligned}$$

and any remaining vertices are assigned the list L_0. Fig. 4.13 illustrates this list assignment for the graph $T(1,21,4)$.

Next, consider $T(1,s,t)$ for $t = (s+1)/4$. Define $\mathcal{L}$ as follows:

$$\begin{aligned} L_1 &: \{s-kt, s-1-kt : k = 0,1,2,3\}, \\ L_2 &: \{s-2-kt, s-3-kt : k = 0,1,2,3\}, \end{aligned}$$

$$L_3 : \{s-4-kt, s-5-kt : k = 0,1,2,3\},$$
$$\cup \{s-6-kt, \dots, s-t+1 : k = 0,1,2\},$$

and any remaining vertices are assigned the list L_0. Fig. 4.14 illustrates this list assignment for the graph $T(1,27,7)$.

Next, consider $T(1,s,t)$ for $t = (s+3)/3$, $s > 27$. Define $\mathcal{L}$ as follows:

$$L_1 : \{s-kt, s-1-kt : k = 0,1,2,3,4\},$$
$$L_2 : \{s-2-kt : k = 1,2,3,4\},$$
$$\cup \{s-3-kt : k = 2,3,4\}$$
$$\cup \{s-4-kt, \dots, s-6-kt : k = 2,3\}$$
$$L_3 : \{1, s-2\} \cup \{s-7-kt, \dots, s-9-kt : k = 1,2,3\}$$
$$\cup \{s-10-kt, \dots, s-12-kt : k = 1,2\}$$
$$\cup \{s-13-kt, \dots, s-t+1-kt : k = 0,1\},$$

and any remaining vertices are assigned the list L_0. Fig. 4.15 illustrates this list assignment for the graph $T(1,45,16)$. The only remaining case for $t = (s+3)/3$ is $T(1,27,10)$, which is isomorphic to $T(1,27,4)$, so this case is completed.

Lastly, consider $T(1,s,t)$ for $t = (s-3)/3$. By the remarks in Section 4.1, we can choose the horizontal normal circuit of $T(1,s,t)$ as the vertical normal circuit to get that $T(1,s,t)$ is isomorphic to a graph $T(1,s,t')$ for some $0 \le t' \le \lfloor (s-1)/2 \rfloor$. A simple calculation shows that either $t' \equiv -(1+t^{-1}) \pmod{s}$ or $t' \equiv t^{-1} \pmod{s}$. Using $s = 3t+3$ and the fact that $t-1 \equiv 0 \equiv s \pmod{3}$, we see that $t' \neq t$. Thus, the graph $T(1,s,t)$ is isomorphic to one of the cases considered earlier, so it is also not 3-choosable.

This covers the 3-chromatic graphs of the form $T(1,s,t)$. The graph $T(2,6,2)$ requires an ad hoc list assignment $\mathcal{L}$ as follows:

$$L_1 : \{(1,1),(1,3),(1,5),(2,1)\}$$
$$L_2 : \{(1,2),(1,4),(1,6),(2,6)\}$$
$$L_3 : \{(2,2),(2,3),(2,4),(2,5)\}.$$

Fig. 4.11 illustrates this list assignment.

The only case left is $T(3,9,3)$, which requires an ad hoc list assignment $\mathcal{L}$ as follows:

$$L_1 : \{(1,1),(1,2),(1,7),(1,8),(2,1),(2,2),(3,1),(3,2)\}$$
$$L_2 : \{(1,3),(1,4),(1,9),(2,3),(2,4),(3,3),(3,4)\}$$

$L_3 : \{(1,5),(1,6),(2,5),(2,6),(2,7),(2,8),(2,9),(3,5),(3,6),(3,7),(3,8),(3,9)\}$.

Fig. 4.12 illustrates this list assignment.

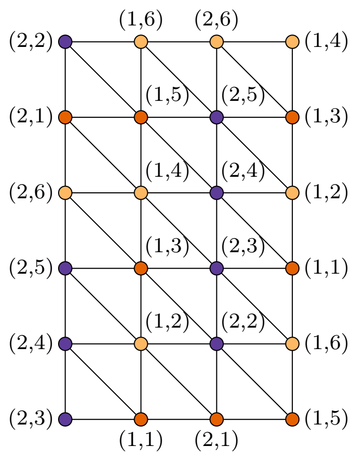


Figure 4.11: A non-$\mathcal{L}$-colorable 3-list-assignment on $T(2,6,2)$

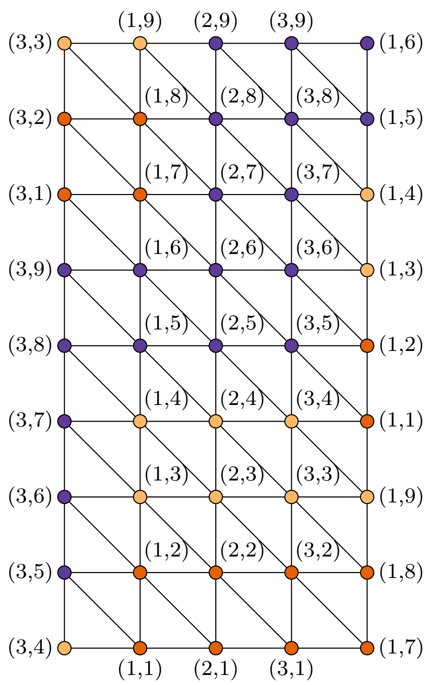
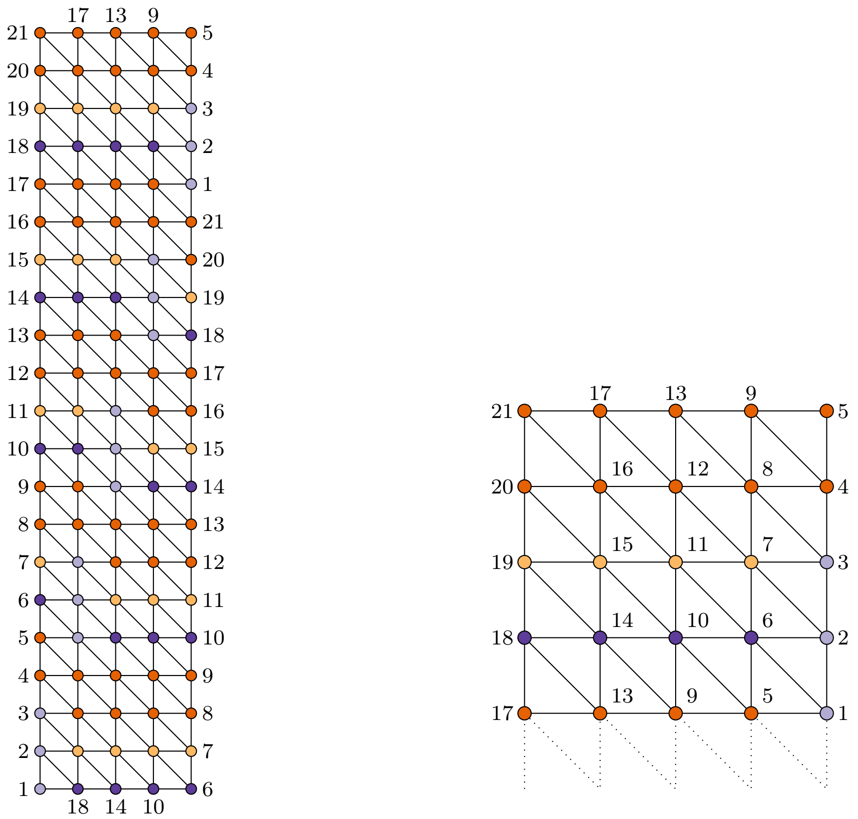
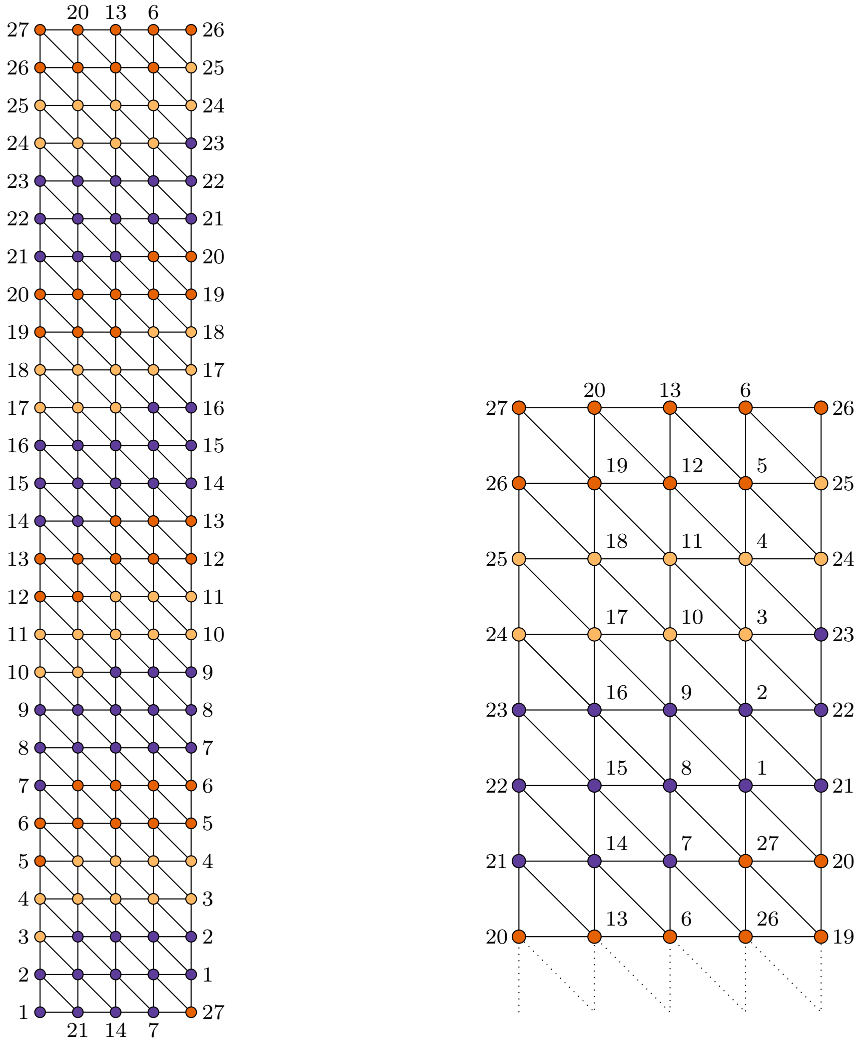


Figure 4.12: A non-$\mathcal{L}$-colorable 3-list-assignment on $T(3,9,3)$



(a) **(b)**

Figure 4.13: A non-$\mathcal{L}$-colorable 3-list-assignment on $T(1, s, 4)$ for $s = 21$



(a) **(b)**

Figure 4.14: A non-$\mathcal{L}$-colorable 3-list-assignment on $T(1, s, (s+1)/4)$ for $s = 27$

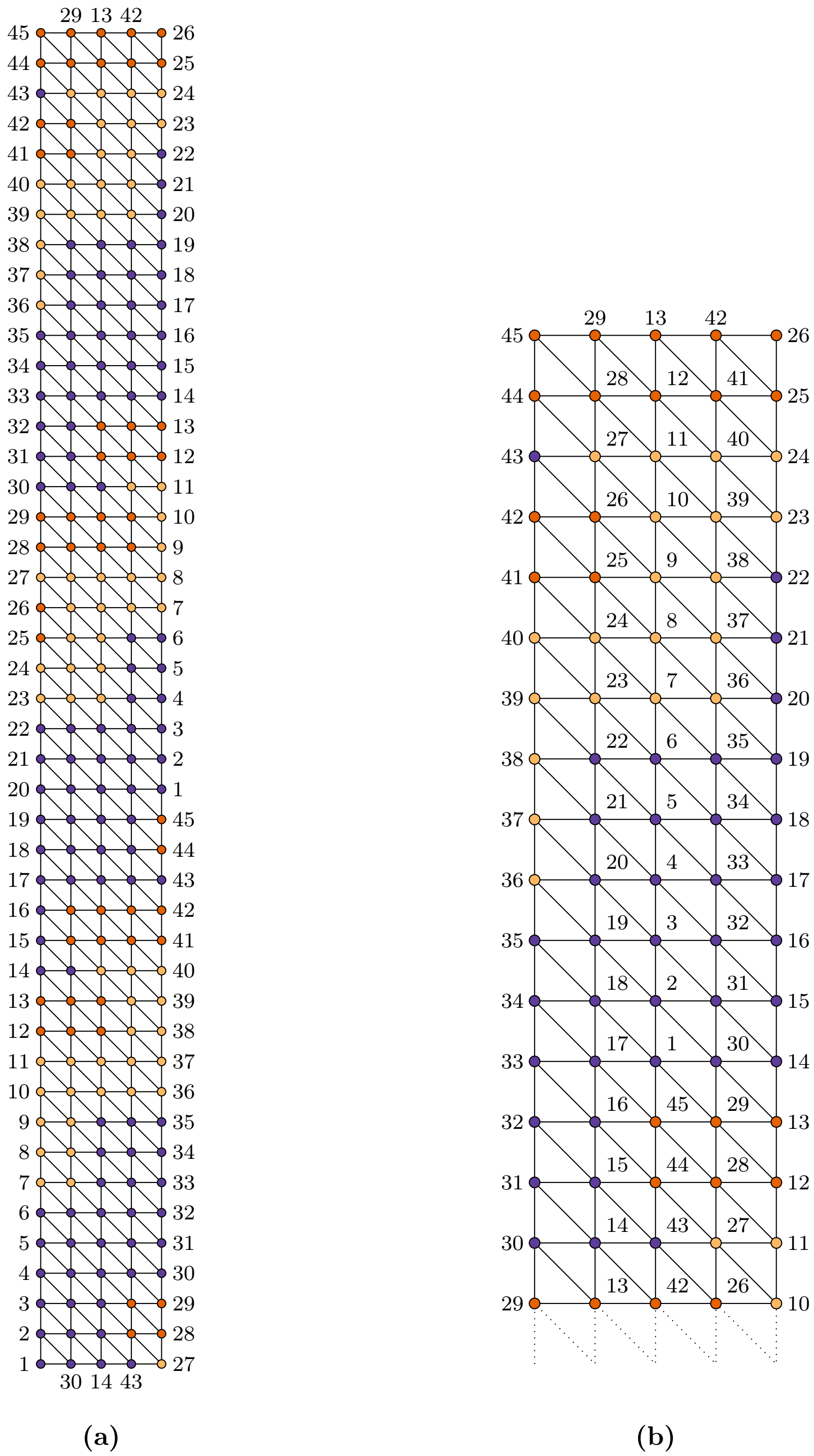


Figure 4.15: A non-$\mathcal{L}$-colorable 3-list-assignment on $T(1, s, (s + 3)/3)$ for $s = 45$

Chapter 5

Coloring the 6-regular toroidal triangulations

5.1 Examining the statement of Theorem 1.12 and its proof

We restate the result of Collins and Hutchinson for convenience:

Theorem 1.12 (Collins–Hutchinson [32, Theorem 1.2]). *Let $G = T(r, s, t)$. If $r, s \geq 3$, then G can be 4-colored, with a finite number of exceptions.*

5.1.1 Constructing counterexamples to Theorem 1.12

Collins and Hutchinson identify that $T(3, 3, 1)$, $T(3, 3, 2)$, $T(3, 5, 3)$, and $T(3, 5, 4)$ are not 4-colorable, but state that there are no others of the form $T(3, s, t)$ for $s \geq 3$ [32, Theorem 3.7]. However, as we show below, the graphs of the form $T(3, s, s-2)$ and $T(3, s, s-1)$ are not 4-colorable for all $s \not\equiv 0 \pmod 4$. Note that the four graphs mentioned in the beginning are obtained by plugging in $s = 3, 5$ in these expressions.

Now, consider the triangulations $T(1, s, 2)$ for $s \geq 7$. These are simple graphs, and, as noted in [32, Section 3], every four successive vertices of $T(1, s, 2)$ induce a K_4. Thus, $T(1, s, 2)$ is 4-colorable for $s \geq 7$ if and only if $s \equiv 0 \pmod 4$.

Therefore, we consider the graphs $T(1, 3s, 2)$ for $s \geq 3$ and $s \not\equiv 0 \pmod 4$. These are all 5-chromatic graphs by [2, 35, 52]. The normal circuits in $T(1, 3s, 2)$ have lengths $3s$, $3s$, and s, respectively, so $T(1, 3s, 2)$ is isomorphic to $T(3, s, t)$ for some $0 \leq t \leq s-1$. Since there are infinitely many $s \geq 3$ such that $s \not\equiv 0 \pmod 4$, there are infinitely many graphs of the form $T(3, s, t)$ that are not 4-colorable, contradicting the statement of Theorem 1.12.

In fact, one can check by a careful computation that $t = s-2, s-1$ in this case. For simplicity, we label the vertex $(1, j)$ with the integer j (recall that j is taken modulo $3s$). Map the vertical, horizontal, and diagonal normal circuits of $T(1, 3s, 2)$ to the horizontal, diagonal, and vertical normal circuits of $T(3, s, 2)$, respectively. Then, the vertical normal circuit of $T(3, s, t)$ has labels $3s, 3s-3, 3s-6, 3s-9, \ldots$ from top to bottom when drawn as in Figure 3.1a. The horizontal normal circuit through the vertex with label $3s$ has the first four labels in the right-to-left direction as $3s$, $3s-1$, $3s-2$, and $3s-3$. Thus, the shift is $t = s-1$. Since $T(r, s, t)$ is isomorphic to $T(r, s, t')$ for $0 \leq t' < s$ such that $t' \equiv -r-t \pmod{s}$, the graph $T(3, s, s-1)$ is isomorphic to $T(3, s, s-2)$. Thus, the graphs $T(3, s, s-1)$ and $T(3, s, s-2)$ are not 4-colorable for all $s \geq 3$ such that $s \not\equiv 0 \pmod{4}$.

5.1.2 Gap in the proof of Theorem 1.12

The proof of Theorem 1.12 in [32] is broken up into a sequence of results, first for the unshifted triangulations $T(r, s, 0)$ [32, Lemma 3.2, Theorem 3.3, Lemma 3.4], and then for the shifted triangulations $T(r, s, t)$ with $t \neq 0$ [32, Theorems 3.6 and 3.7]. We identify the following theorem as the source of the contradiction:

Theorem 5.1 (Collins–Hutchinson [32, Theorem 3.6])**.** *Let $G = T(r, s, t)$ for some $0 < t \leq s-1$. Then if $3 \leq s, r$, G can be 4-colored except possibly in the case when $r = 5$, or when $t = s-1$ and $r = s$ or $s+1$, or when $t = s-2$ and $r = s$.*

The proof of this theorem proceeds as follows. Let C_i denote the ith column of $T(r, s, t)$, for $i = 1, \ldots, r$. First, a proper 4-coloring of $T(y, s, 0)$ is used to color $C_1, \ldots, C_y$, where $y \geq 3$ is to be determined. (Note that $y \geq 3$ is needed to ensure that a proper 4-coloring of $T(y, s, 0)$ can be found using [32, Theorem 3.3].) Then, the coloring on C_1 is repeated on C_{y+1}. Then, the coloring on C_{y+1} is repeated on C_{y+2} after an upward shift by one vertex. As shown in [32, Lemma 3.1], this coloring on C_{y+2} is compatible with the coloring on C_{y+1} as long as $r, s \neq 5$. Continue this process of repeating the coloring on successive columns with an upward shift to color the t columns $C_{y+2}, \ldots, C_{y+t+1}$. Now, note that the coloring on C_{y+t+1} is identical to the coloring on C_1, except that it is shifted upwards by t vertices. Thus, this gives a valid coloring of $T(r, s, t)$ provided that $r = y + t$.

At this point, Collins and Hutchinson state that the inequality $y \geq 3$ fails only when $t = s-1$ and $r = s$ or $s+1$, or when $t = s-2$ and $r = s$, so this concludes their proof. However, this conclusion can only be drawn under the additional hypothesis that $r \geq s$. Thus, their proof holds only under the additional hypothesis that $r \geq s$.

Hence, the statements of Theorem 5.1 and [32, Theorem 3.7] need to be modified by adding the hypothesis that $r \geq s$. However, the statement of [32, Theorem 3.8] is now weakened, since the colorability of the triangulations $T(2, s, t)$ (for odd s) that are not isomorphic to $T(1, 2s, t')$ for any $0 \leq t' < 2s$ is no longer completely settled by their previous results.

Furthermore, the above method of proof does not seem to easily lend itself to the cases when $r < s$. The above argument does extend to the graphs $T(r, s, t)$, $3 \leq r, s$, $r \neq 5$, $s \neq 5$, with $r \geq t + 3$ or $r > s - \lceil t/2 \rceil$, the latter by extending the coloring on C_y by using downward shifts by two vertices instead of upward shifts by one vertex, but it is not clear, for instance, how one can 4-color the graph $T(10, 990, 100)$ by an argument along the above lines.

In the next section, we provide a different argument to color the shifted triangulations.

5.2 Main result

We shall use Collins and Hutchinson's coloring of the unshifted triangulations [32, Theorem 3.3] to color the shifted triangulations.

Theorem 5.2. *Let $G = T(r, s, t)$ be a simple triangulation with $r \neq 1$ and s as the maximal length of a normal circuit in G. Then G is 4-colorable.*

Proof. Note that the conditions on r and s imply that $\gcd(s, t), \gcd(s, r + t) \geq r \geq 2$.

Suppose that $r \geq 3$. Now, use a proper coloring of $T(r, \gcd(s, t), 0)$ to color the first $\gcd(s, t)$-many rows of $T(r, s, t)$, and repeat this coloring block $(s/\gcd(s, t))$-many times to get a coloring of $T(r, s, t)$. For this coloring to be proper, the coloring on the column C_r is required to be compatible with an upward shift by t vertices of the coloring on the column C_1. But, since the coloring on the column C_1 is periodic with period $\gcd(s, t)$, an upward shift by t vertices of the coloring on C_1 is the same as no shift. Thus, we only need to check that the coloring on C_r is compatible with that on C_1, and this holds since it is obtained from a periodic coloring of the unshifted triangulation $T(r, \gcd(s, t), 0)$.

Next, suppose that $r = 2$. As observed in [32, Theorem 3.8], if s is even, then G is 4-colorable, simply by 2-coloring the columns C_1 and C_2 with the colors $\{1, 2\}$ and $\{3, 4\}$, respectively.

So, suppose that $r = 2$ and that s is odd, which imply that $\gcd(s, t), \gcd(s, t + 2) \geq 3$. Thus, G is isomorphic to $T(r', s', t')$ for $r' = \gcd(s, t)$, $s' = 2s/\gcd(s, t)$, and some $0 \leq t' < s'$ such that $\gcd(s', t') = \gcd(s, t + 2) \geq 3$. This is possible by the remarks in Section 2.4. Thus, we can repeat the previous algorithm to color $T(r', s', t')$ as follows. First, use a proper coloring of $T(r', \gcd(s', t'), 0)$ to color the first $\gcd(s', t')$-many rows of

$T(r', s', t')$. Then, repeat this coloring block $(s'/\gcd(s', t'))$-many times to get a coloring of $T(r', s', t')$. This is verified to be a proper coloring for the same reason as in the case $r \geq 3$, so this completes the case $r = 2$ as well. $\square$

In fact, the proof in the case $r = 2$ and s odd shows that the following theorem is also true.

Theorem 5.3. *Let $G = (V, E)$ be a simple 6-regular triangulation on the torus with normal circuits of lengths $a \geq b \geq c$ such that $\frac{n}{c} \geq \frac{n}{b} \geq 3$, where $|V| = n$. Then, G is 4-colorable.*

Specifically, a coloring of G can be found by viewing G as $T\left(\frac{n}{c}, c, t\right)$, where $0 \leq t < c$ is such that $\gcd(c, t) = \frac{n}{b}$, and then coloring G by repeating $(c/\gcd(c, t))$-many times a proper coloring of $T\left(\frac{n}{c}, \frac{n}{b}, 0\right)$. This can be done since it is assumed that $\frac{n}{c} \geq \frac{n}{b} \geq 3$.

5.2.1 Proof of Theorem 1.13

We restate the theorem below for convenience:

Theorem 1.13. *Let $G = T(r, s, t)$ be a simple 6-regular triangulation having normal circuits of lengths $a \geq b \geq c$. Suppose that $\left(\frac{n}{a}, \frac{n}{b}\right) \neq (1, 1), (1, 2)$, where $n = rs$ is the order of G. Then G can be 4-colored.*

Proof. Suppose that $G = T(r, s, t)$ is a simple 6-regular triangulation on the torus with normal circuits of lengths $a \geq b \geq c$ such that $(\frac{n}{a}, \frac{n}{b}) \neq (1, 1), (1, 2)$, where $n = rs$.

If $\frac{n}{a} = 1$, then $3 \leq \frac{n}{b} \leq \frac{n}{c}$, so G is 4-colorable by Theorem 5.3.

If $\frac{n}{a} = 2$, then G is isomorphic to $T(2, a, t)$ for some $0 \leq t < a$. If $\frac{n}{b} = 2$, then a is even, since $b = n/\gcd(a, t)$ or $n/\gcd(a, t + 2)$. So G is 4-colorable by Theorem 5.2. If $\frac{n}{b} \geq 3$, then $\frac{n}{c} \geq 3$, so again we are done by Theorem 5.3.

If $\frac{n}{a} \geq 3$, then G is isomorphic to $T(r', a, t)$ for some $0 \leq t < a$, where $r' \geq 3$. So, we are done by Theorem 5.2. $\square$

5.3 Summary of the colorability of 6-regular toroidal triangulations

In this section, we shall present a complete picture of the colorings of 6-regular triangulations on the torus as in Theorem 1.14.

By the results in Section 5.2, we are left to classify the colorability of those 6-regular triangulations G that are either loopless multigraphs, or isomorphic to some simple $T(r, s, t)$ having normal circuits of lengths $a \geq b \geq c$ such that $\left(\frac{n}{a}, \frac{n}{b}\right) = (1, 1)$ or $(1, 2)$.

5.3.1 The loopless multigraphs of the form $T(1, s, t)$

Note that the graphs $T(1, s, t)$ and $T(1, s, s - t - 1)$ are isomorphic (by the remarks in Section 2.4). So, when $r = 1$ we shall only focus on the values of t in the range $0 \leq t \leq \lfloor (s-1)/2 \rfloor$.

Now, it is easy to check that $T(1, s, t)$ has loops if and only if either $s \leq 2$ or $t = 0$, and that $T(1, s, t)$ is a loopless multigraph if and only if $s \geq 3$ and $t = 1, \lfloor (s-1)/2 \rfloor$.

So, we start by considering the graph $T(1, s, 1)$ for $s \geq 3$. Collins and Hutchinson [32] gave explicit 4-colorings of $T(1, s, 1)$ for $s > 5$. Furthermore, Yeh and Zhu [104, Theorem 6] observed that $T(1, s, 1)$ is 3-chromatic if and only if $s \equiv 0 \pmod 3$ (after deleting the duplicated edges in $T(1, s, 1)$, this graph is isomorphic to $G_s[1, 2]$ in their notation). Lastly, one can see that the graph $T(1, 5, 1)$ is isomorphic to K_5 after deleting the duplicated edges, so it is 5-chromatic.

Next, we consider the graph $T(1, s, \lfloor (s-1)/2 \rfloor)$ for $s \geq 5$. For $s = 2k + 1$ ($k \geq 2$), Yeh and Zhu [104, Theorem 6] have shown that this graph is isomorphic to $T(1, s, 1)$ (after deleting the duplicated edges in $T(1, s, k)$ for $s = 2k + 1$, this graph is isomorphic to the graph $G_s[1, k]$ in their notation). Hence, $T(1, 2k+1, k)$ is 4-colorable for all $k \geq 3$, and is 3-chromatic if and only if $s \equiv 0 \equiv k - 1 \pmod 3$, and $T(1, 5, 2)$ is 5-chromatic since it is isomorphic to K_5 after deleting the duplicated edges.

For $s = 2k+2$ ($k \geq 2$), Yeh and Zhu [104, Theorem 5] have shown that $T(1, s, \lfloor (s-1) \rfloor /2)$ is 4-colorable (and in fact 4-chromatic) if and only if $s \equiv 0 \pmod 4$ (this graph is isomorphic to $G_s[1, k, k + 1]$ in their notation). In this case, by removing the duplicated edges we get a 5-regular graph on the torus. So, by Brooks's theorem (Theorem 3.4), when $k \geq 4$ is even the graph $T(1, 2k + 2, k)$ is 5-chromatic, and when $k = 2$ the graph $T(1, 6, 2)$ is isomorphic to K_6 after deleting the duplicated edges, and hence is 6-chromatic.

5.3.2 The loopless multigraphs of the form $T(2, s, t)$

The graph $T(2, s, t)$ has loops if and only if $s = 1$, so we assume that $s \geq 2$. One can check that $T(2, s, t)$ is a loopless multigraph if and only if $t = 0$, $s - 2$, or $s - 1$. Furthermore, $T(2, s, 0)$ and $T(2, s, s - 2)$ are isomorphic (see the remarks in Section 2.4), so there are only two cases to consider.

As observed in [32, Theorem 3.8], $T(2, s, 0)$ is 4-colorable (and in fact 4-chromatic) if and only if $s \geq 2$ is even. When $s \geq 3$ is odd, $T(2, s, 0)$ is isomorphic to $T(1, 2s, \lfloor (s-1)/2 \rfloor)$, which was discussed earlier. Next, we look at $T(2, s, s-1)$. This graph is isomorphic to $T(1, 2s, 1)$ for all $s \geq 2$, which we have discussed earlier. So, this completes the case $r = 2$.

5.3.3 The loopless multigraphs of the form $T(r, s, t)$ for $r \geq 3$

The graph $T(r, s, t)$ for $r \geq 3$ has loops if and only if $s = 1$, and it is a loopless multigraph if and only if $s = 2$. When $t = 0$, the graph $T(r, 2, 0)$ is isomorphic to $T(2, r, 0)$, which we have discussed earlier. When $t = 1$, the graph $T(r, 2, 1)$ is isomorphic to $T(1, 2r, \lfloor (r-1)/2 \rfloor)$, which was also discussed earlier.

Thus, the colorability of all the loopless multigraphs $T(r, s, t)$ is known. Next, we need to consider the colorability of the simple graphs $T(r, s, t)$. Theorem 1.13 covers the 4-colorability of those $T(r, s, t)$ that have normal circuits of lengths $a \geq b \geq c$ such that $\left(\frac{n}{a}, \frac{n}{b}\right) \neq (1, 1), (1, 2)$, where $n = rs$. So, we are only left to consider the remaining cases, namely when $\left(\frac{n}{a}, \frac{n}{b}\right) = (1, 1)$ or $(1, 2)$. As a step towards that, let us first consider the colorability of the simple graphs of the form $T(1, s, t)$.

5.3.4 The simple graphs $T(1, s, t)$

From the previous discussions, it suffices to consider the graphs $T(1, s, t)$ for those values of t in the range $2 \leq t \leq \lfloor (s-1)/2 \rfloor - 1$. In particular, we assume that $s \geq 7$ in what follows.

Now, as shown in [32, Theorem 3.8] and discussed above in Section 5.1.1, the graphs $T(1, s, 2)$ are simple triangulations that are 4-colorable (and in fact 4-chromatic) if and only if $s \equiv 0 \pmod 4$ since every four consecutive vertices in $T(1, s, 2)$ induce a K_4. Collins and Hutchinson [32] observe that these grids are all easily seen to be 5-chromatic when $s \geq 15$. Explicit 5-colorings for all $s \geq 8$, $s \neq 11$, in the spirit of Collins and Hutchinson's work, can be given as follows: write $s = 4u + 5v$ for $u \geq 0$ and $v \in \{0, 1, 2, 3, 4\}$ (which can be done for all $s \geq 8$, $s \neq 11$), and color $T(1, s, 2)$ using u sets of 1234 followed by v sets of 12345. This is easily seen to be a proper coloring of $T(1, s, 2)$.

When $s = 11$, the coloring 12345123456 is seen to work: this is the 6-chromatic graph on 11 vertices found by Albertson and Hutchinson [2], which is also the unique simple 6-regular triangulation on the torus having 11 vertices, up to isomorphism.

When $s = 7$, $T(1, 7, 2)$ is 7-chromatic since it is isomorphic to K_7.

Next, for each $t \geq 3$, Collins and Hutchinson [32, Theorem 3.9] exhibited 4-colorings for all but finitely many of the graphs $T(1, s, t)$ with s such that $t \leq \lfloor (s-1)/2 \rfloor - 1$. The remaining cases were handled by Yeh and Zhu [104, Theorem 5]:

Theorem 5.4 (Yeh–Zhu [104, Theorem 5], 2003)**.** *Let $G = T(1, s, t)$ be a simple triangulation on the torus, for $3 \leq t \leq \lfloor (s-1)/2 \rfloor - 1$. Then G is 4-colorable, unless G satisfies one of the following conditions:*

1. $s \in \{2t+3,\ 3t+1,\ 3t+2\}$ *and* $s \not\equiv 0 \pmod 4$*; or*
2. $(s, t) \in \{(13, 3),\ (17, 3),\ (17, 4),\ (17, 6),\ (18, 3),\ (19, 7),\ (25, 3),\ (25, 6),\ (25, 7),\ (25, 9),\ (25, 10),\ (26, 7),\ (26, 10),\ (33, 6),\ (33, 14),\ (37, 10)\}$.

Yeh and Zhu have also shown that the graphs $T(1, s, t)$ for $s \in \{2t+3,\ 3t+1,\ 3t+2\}$ are in fact isomorphic to $T(1, s, 2)$. Note that $T(1, s, t)$ is isomorphic to the graph $G_s[1, t, t+1]$ in their notation.

5.3.5 The simple graphs $T(r, s, t)$ with $\left(\frac{n}{a}, \frac{n}{b}\right) = (1, 1)$ or $(1, 2)$

Let $G = (V, E)$ be a simple 6-regular triangulation on the torus with $|V| = n$. Suppose that G has normal circuits of lengths $a \geq b \geq c$ such that $\left(\frac{n}{a}, \frac{n}{b}\right) = (1, 1)$ or $(1, 2)$. Then, G can be represented as $T(1, s, t)$, and by the discussion in Section 5.3.4 we know exactly what values t can take if G is 5-chromatic. Thus, to classify the 5-chromatic graphs G satisfying $\left(\frac{n}{a}, \frac{n}{b}\right) = (1, 1)$ or $(1, 2)$, it suffices to consider the 5-chromatic graphs of the form $T(1, s, t)$ discussed in Section 5.3.4 and see whether and how they can be represented as $T(r', s', t')$ with $r' > 1$.

First, consider the graphs $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$, $s \geq 9$, $s \neq 11$. Since its normal circuits have lengths s, $s/\gcd(s, 2)$, and $s/\gcd(s, 3)$, it can be represented as $T(r', s', t')$ with $r' > 1$ only if s is a multiple of 2 or 3. The chromaticity of the graphs $T(1, 3s, 2)$ was discussed in Section 5.1, and a similar analysis can be done for the graphs $T(1, 2s, 2)$ with $s \not\equiv 0 \pmod 2$ to show that $T(2, s, 1)$ and $T(2, s, s-3)$ are 5-chromatic for all odd $s \geq 5$.

Next, we consider the exceptional graphs listed in Theorem 5.4. The graphs listed in the first point in Theorem 5.4 are already covered by the above analysis, since Yeh and Zhu have shown that the graphs $T(1, s, t)$ for $s \in \{2t+3,\ 3t+1,\ 3t+2\}$, $s \not\equiv 0 \pmod 4$, $t \geq 3$, are all isomorphic to $T(1, s, 2)$.

Thus, it only remains to consider the finitely many exceptional graphs listed in the second point in Theorem 5.4 that have composite order. A similar analysis can be done for these graphs as was done in Section 5.1 for $T(1, 3s, 2)$. We omit the details and only state the final results in the next theorem. Just one observation needs to be added before we do so:

it is easy to show that a simple graph $T(r,s,t)$ is 3-chromatic if and only if $s \equiv 0 \equiv r-t \pmod 3$.

We are now ready to compile the above results along with the known results from the previous work of [2, 32, 35, 52, 104] to characterize the colorability of all the 6-regular toroidal triangulations. We follow the convention as adopted in [32, 104] to specify the classification by the parameters r, s, and t, instead of only listing isomorphism classes of the graphs.

Theorem 1.14. *Let $G = T(r,s,t)$ for $r \geq 1$, $s \geq 1$, $0 \leq t \leq s-1$ be a 6-regular triangulation on the torus. If $r = 1$, then $T(1,s,t)$ is isomorphic to $T(1,s,s-t-1)$, so in this case consider t only in the range $0 \leq t \leq \lfloor (s-1)/2 \rfloor$.*

1. *G contains loops if and only if either $s = 1$, or $r = 1$ and $s = 2$, or $r = 1$ and $t = 0$.*
2. *G is 7-chromatic if and only if G is isomorphic to K_7, and this happens only when $G = T(1,7,2)$.*
3. *G is 6-chromatic if and only if G is isomorphic either to K_6 (after deleting duplicated edges), or to the graph of Albertson and Hutchinson [2] on 11 vertices. The former happens only when $G \in \{T(1,6,2),\ T(2,3,0),\ T(2,3,1),\ T(3,2,0),\ T(3,2,1)\}$ and the latter only when $G \in \{T(1,11,2),\ T(1,11,3),\ T(1,11,4)\}$.*
4. *G is 5-chromatic if and only if G is one of the following graphs:*
   (a) *$T(1,5,1)$, $T(1,5,2)$ (these are isomorphic to K_5 after deleting duplicated edges);*
   (b) *$T(1,s,2)$ for $s \geq 9$, $s \neq 11$, $s \not\equiv 0 \pmod 4$;*
   (c) *$T(1,s,t)$ for $s \in \{2t+2,\ 2t+3,\ 3t+1,\ 3t+2\}$, $s \geq 9$, $s \not\equiv 0 \pmod 4$;*
   (d) *$T(2,s,0)$, $T(2,s,1)$, $T(2,s,s-3)$, $T(2,s,s-2)$ for odd $s \geq 5$;*
   (e) *$T(3,s,s-2)$, $T(3,s,s-1)$ for $s \geq 3$, $s \not\equiv 0 \pmod 4$;*
   (f) *$T(r,2,0)$, $T(r,2,1)$ for odd $r \geq 5$;*
   (g) *$T(1,s,t)$ for $(s,t) \in \{(13,3),\ (17,3),\ (17,4),\ (17,6),\ (18,3),\ (19,7),\ (25,3),\ (25,6),\ (25,7),\ (25,9),\ (25,10),\ (26,7),\ (26,10),\ (33,6),\ (33,14),\ (37,10)\}$;*
   (h) *$T(2,s,t)$ for $(s,t) \in \{(9,3),\ (9,4),\ (13,3),\ (13,8)\}$;*
   (i) *$T(3,s,t)$ for $(s,t) \in \{(6,1),\ (6,2),\ (11,2),\ (11,6)\}$;*
   (j) *$T(5,s,t)$ for $(s,t) \in \{(5,2),\ (5,3)\}$.*
5. *G is 4-colorable in all other cases.*
6. *In particular, G is 3-chromatic if and only if $s \equiv 0 \equiv r-t \pmod 3$.*

Part II

Extremal set theory

Chapter 6

Introduction

6.1 Fractional intersecting families

The theory of set systems with restricted intersection sizes is a classical and well-studied problem and the basic template of the problem is as follows.

> Given a set L of non-negative integers, determine the maximum size of a family $\mathcal{F} \subseteq \mathcal{P}([n])$ of subsets of $[n] \coloneqq \{1, \ldots, n\}$ such that for distinct $A, B \in \mathcal{F}$ we have $|A \cap B| \in L$.

This problem has its origins in the de Bruijn–Erdős theorem [24] with further extensions including the Ray-Chaudhuri–Wilson inequality [82], the Frankl–Wilson inequality [46], and the Alon–Babai–Suzuki inequality [8] among a host of other interesting results [44, 81, 90] and has spawned several variants, each with its own set of highlights and difficulties besides ushering in a wide range of combinatorial and algebraic tools that are now an integral component of combinatorial techniques for extremal problems.

A recent variant [17] of this problem, which is the principal focus of this part of the thesis, introduces the notion of *fractional intersecting families* which goes as follows.

> Suppose $L \coloneqq \{\theta_1, \ldots, \theta_\ell\}$ is a set of proper positive fractions, that is, $0 < \theta_i = \frac{a_i}{b_i} < 1$ and $\gcd(a_i, b_i) = 1$ for each i. Say that $\mathcal{F} \subseteq \mathcal{P}([n])$ is a *fractional L-intersecting family* (or that $\mathcal{F}$ is *fractionally L-intersecting*) if, for any two distinct sets $A, B \in \mathcal{F}$, there exists $\theta \in L$ such that $|A \cap B| \in \{\theta|A|, \theta|B|\}$.
>
> **Question.** *How large can a fractional L-intersecting family be?*

This problem remains unresolved; the best known bounds are a poly-logarithmic factor away from optimal bounds [17]. In this thesis, I shall focus on the case when $|L| = 1$, so henceforth we shall always have $L = \{\theta\}$, where $\theta = \frac{a}{b}$ is a proper positive fraction with $\gcd(a,b)=1$, and we shall also use the term "θ-intersecting" interchangeably with "L-intersecting".

Now, one of the main results in [17] states that if $\mathcal{F}$ is a fractional θ-intersecting family, then $|\mathcal{F}| \le O_b(n\log n)$ (that is, $|\mathcal{F}| \le Cn\log n$ for all sufficiently large n, where the constant C depends on b). On the lower bound side, there are constructions of fractional θ-intersecting families of size $\Omega(n)$. For $\theta=\frac{1}{2}$, one can improve upon the constant a little more; there exist bisection closed families[1] of size $\lfloor\frac{3n}{2}\rfloor-2$. What makes the problem of determining the size of maximal bisection closed families more interesting and intriguing is that there are non-isomorphic families of size $\lfloor\frac{3n}{2}\rfloor-2$. The simplest example (and an instructive one at that) is the following.

Example 6.1. For the sake of simplicity, denote the set $\{x_1,\dots,x_\ell\}$ by $x_1\cdots x_\ell$. Consider the family

$$\mathcal{F}=\begin{cases}\{12,13,\dots,1n,1234,1256,\dots,12(n-1)n\}, & n\equiv 0 \pmod 2;\\ \{12,13,\dots,1n,1234,1256,\dots,12(n-2)(n-1)\}, & n\equiv 1 \pmod 2.\end{cases}$$

Then, $\mathcal{F}$ is a bisection closed family over $[n]$.

The other known bisection closed families of size $\lfloor\frac{3n}{2}\rfloor-2$ arise from a construction using Hadamard matrices.

Example 6.2. Let H be an $m\times m$ Hadamard matrix, i.e. a matrix whose entries lie in $\{\pm1\}$, and with all the rows being mutually orthogonal. Assume that H is normalized so that the first row is the all-ones vector. Let J denote the $m\times m$ all-ones matrix. Consider the matrix

$$\begin{bmatrix} H & H\\ H & -H\\ H & -J\end{bmatrix},$$

and delete the first and $(2m+1)$th rows. Viewing the remaining rows as the $\{\pm1\}$ incidence vectors of subsets of $[2m]$, one can verify that this defines a family $\mathcal{F}\subset\mathcal{P}([n])$ that is 2-bisection closed, where $n=2m$. Since there are $3m-2$ sets in $\mathcal{F}$, we have $|\mathcal{F}|=\frac{3n}{2}-2$.

6.1.1 Hierarchically intersecting families

One of the principal reasons why a linear bound, let alone a tight bound, for the size of a bisection closed family is elusive is this diffusive nature of the known families of maximal

[1]When $\theta=1/2$ a fractional θ-intersecting family is also called a bisection closed family in [17].

size. In fact, these two examples are at the extreme ends of a tower of *hierarchically* bisection closed families. Before explaining further, let us observe that the first example satisfies the following stronger property: for any $r \geq 2$ and any sets $A_1, \ldots, A_r \in \mathcal{F}$ we also have $|A_1 \cap \cdots \cap A_r| \in \{\frac{1}{2}|A_1|, \ldots, \frac{1}{2}|A_r|\}$. The easiest way to see this is to note that for this family, the subfamilies of sizes 2 and 4 are sunflowers, and also that any collection of subsets in $\mathcal{F}$ have non-empty intersection. On the other hand, the second example does not satisfy this stronger property for any $r > 2$, which shows that it is structurally very different from the other. This raises the following more natural question: how large could a hierarchically bisection closed family be?

In order to make this precise, we make a formal definition.

Definition 6.3. Let $r \geq 2$ and $L = \{\theta_1, \ldots, \theta_m\}$ be a set of fractions in $(0, 1)$. A family $\mathcal{F}$ of subsets of $[n]$ is called *hierarchically r-closed L-intersecting* (or simply *r-closed L-intersecting*) if, for each $2 \leq t \leq r$ and any t distinct sets $A_1, \ldots, A_t$ in $\mathcal{F}$ we have $|\bigcap_{i=1}^{t} A_i| \in \{\theta_j |A_i| : 1 \leq i \leq t,\ 1 \leq j \leq m\}$.

When $L = \{\theta\}$, an r-closed L-intersecting family is also called an *r-closed θ-intersecting* family. In particular, when $\theta = 1/2$, we call such a family *r-bisection closed*.

Note that if a θ-intersecting family is r-closed, then it is also s-closed for all $2 \leq s \leq r$, so that explains why we refer to such a family as hierarchically closed.

The natural question that arises is the following. Suppose $r \geq 3$. If $\mathcal{F} \subset \mathcal{P}([n])$ is r-closed θ-intersecting, then determine the optimal upper bound for $|\mathcal{F}|$. Note that if $r = 2$, then we are back to the case of fractional L-intersecting families, so it behooves us to set $r \geq 3$ if we hope to see any different emergent phenomenon arising from the definition. And the Chapter 7 of this thesis shall attest that setting $r \geq 3$ makes a big difference.

It is imperative to compare this notion with another generalization that appears in [71] which goes as follows. For an integer $r \geq 2$, and L as above, a family $\mathcal{F}$ is said to be *r-wise fractionally L-intersecting* if for any distinct $A_1, \ldots, A_r \in \mathcal{F}$ there exists $\theta \in L$ such that $|A_1 \cap \cdots \cap A_r| \in \{\theta|A_1|, \ldots, \theta|A_r|\}$. Again, the problem of determining the size of a maximum r-wise fractional L-intersecting family is optimally determined in [71] up to poly-logarithmic factors, and it appears that to get beyond the poly-logarithmic factor needs newer ideas (see [17] for more details on this). Our notion of r-closed θ-intersecting is somewhat related and yet vastly different as the main results of the next chapter will attest.

We are now in a position to state the main results of the next chapter.

Theorem 6.4. *Let $\mathcal{F}$ be an r-closed θ-intersecting family over $[n]$, with $r \geq 3$. Let $\theta = a/b \in (0, 1)$ with* $\gcd(a, b) = 1$, $a, b > 0$.

1. *If $a > 1$, then $|\mathcal{F}| \leq 2\left(\frac{\ln(b)-\ln(a)+1}{b-a}\right)(n-a)+1$.*
2. *If $a = 1$, then we have two cases:*
   - (a) *if $b = 2$ and $\mathcal{F}$ contains a set of size 2, then $|\mathcal{F}| \leq (1+\ln(2))(n-1)+1$;*
   - (b) *otherwise, $|\mathcal{F}| \leq \left(\frac{2\ln(b)}{b-1}\right)(n-1)+1$.*

As mentioned earlier, the case $\theta = 1/2$ is of particular interest, and in this case, we have a tight upper bound. In fact we prove something substantially stronger.

Theorem 6.5. *Let $\mathcal{F}$ be an r-bisection closed family over $[n]$, with $r \geq 3$. Then,*

$$|\mathcal{F}| \leq \lfloor \tfrac{3n}{2} \rfloor - 2 \qquad (*)$$

for all $n \geq 2$. Moreover:

1. *(Tightness) For each $n \geq 2$, there exists an r-bisection closed family $\mathcal{F}_{\max}$ over $[n]$ which attains the above bound.*
2. *(Uniqueness) For any family $\mathcal{F}$ over $[n]$ that attains the above bound, there is a permutation σ of $[n]$ such that $\mathcal{F}_{\max} = \sigma(\mathcal{F}) := \{\sigma(A) : A \in \mathcal{F}\}$, where $\sigma(E) := \{\sigma(a) : a \in E\}$ for any set $E \in \mathcal{P}([n])$.*
3. *(Stability) There exists an absolute constant $C > 0$ such that the following holds. If $|\mathcal{F}| \geq (\frac{3}{2} - \epsilon)n$ for some $0 < \epsilon < 0.1$, then for some permutation σ of $[n]$,*

$$|\sigma(\mathcal{F}) \setminus \mathcal{F}_{\max}| < C\epsilon n.$$

6.1.2 The minimum rank problem for generalized adjacency matrices

Let us return to the study of fractional intersecting families to view it from a linear algebraic perspective, as suggested in [17]. In doing so, the authors posed a more general question concerning the ranks of certain families of matrices. Indeed, the problem of determining the ranks of specific matrices has generally been of immense interest in extremal combinatorics with applications in theoretical computer science as well—see [7, 20, 25, 30, 41, 54, 60, 83]. This problem, motivated by the study of fractional intersecting families, is another fruitful walk down the same path.

To describe the problem, we introduce a small amount of notation. Let $\mathbb{F}$ denote an arbitrary field. At the moment, we do not place any restriction on the characteristic of the field; any such requirement will be made explicit when the need arises. Suppose that $\mathbf{a} := (a_1, a_2, a_3, \dots)$ is a sequence in $\mathbb{F}$ and $f : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is a function such that

$f(a_i, a_i) \neq 0$ for all i. Let $\mathbf{a}_n$ denote the truncated (finite) sequence $(a_1, \ldots, a_n)$. Let $\mathcal{M}_n^{(f)}(\mathbf{a})$ consist of the family of all symmetric $n \times n$ matrices over $\mathbb{F}$ with all diagonal entries being zero and such that for $1 \leq i < j \leq n$ the (i,j)th entry is either $f(a_i, a_j)$ or $f(a_j, a_i)$. Let $\text{rank}(M)$ denote the rank of M over the field $\mathbb{F}$.

Problem 6.6. Determine $\min\{\text{rank}(M) : M \in \mathcal{M}_n^{(f)}(\mathbf{a})\}$.

This problem was formulated for the function $f(x,y) = x$ and a sequence $\mathbf{a}$ of nonzero elements in the field $\mathbb{F} = \mathbb{R}$ in [17], where the following question was asked:

Question 6.7. *Is there is an absolute constant $c > 0$ such that $\text{rank}(M) \geq cn$ for all $M \in \mathcal{M}_n^{(f)}(\mathbf{a})$?*

An affirmative answer to the above question for a general linear function $f(x,y) = \alpha x + \beta y$ would in fact show that any θ-intersecting family has size $O(n)$, and we include that simple argument here for the sake of completeness. Suppose $\mathcal{F}$ is a θ-intersecting family over $[n]$ of size m and let $X_{m \times n}$ be the matrix whose rows are indexed by the members of $\mathcal{F}$ and the columns by the elements of $[n]$ defined as follows: for $A \in \mathcal{F}$ and $x \in [n]$, set $X(A,x) := 1$ if $x \in A$ and $X(A,x) := -1$ otherwise. Then the matrix XX^T whose rows and columns are indexed by the members of $\mathcal{F}$ satisfies

$$\begin{aligned} XX^T(A,A) &= n, \\ XX^T(A,B) &= n - 2(|A| + |B|) + 4|A \cap B| \\ &= \begin{cases} n - 2|A| - 2(1-2\theta)|B|, & |A \cap B| = \theta|B|; \\ n - 2|B| - 2(1-2\theta)|A|, & |A \cap B| = \theta|A|. \end{cases} \end{aligned}$$

In particular, if $\mathcal{F} = \{A_1, \ldots, A_m\}$ and we let J denote the $m \times m$ matrix consisting entirely of ones, then $\frac{1}{2}(nJ - XX^T) \in \mathcal{M}_n^{(f)}(\mathbf{a})$ where $\mathbf{a}_n = (|A_1|, \ldots, |A_m|)$ and $f(x,y) = x + (1-2\theta)y$. Hence, if there is an affirmative answer to Question 6.7, then $\text{rank}(XX^T) \geq cm$. But, since $\text{rank}(XX^T) \leq \text{rank}(X) \leq n$, it follows that $m \leq (n+1)/c$, which establishes an asymptotically tight bound on the size of the bisection closed family.

The notion of bisection-closed families over $[n]$ (more generally, of *fractional* L-intersecting families) has been generalized in other directions. For instance, in [68] the authors consider a fractional variant of ℓ-cross-intersecting pairs of families in $[n]$. They characterize the maximal $\frac{c}{d}$-cross-intersecting pairs, and in particular the maximal cross-bisecting pairs (i.e., when $\frac{c}{d} = \frac{1}{2}$). In [67], the authors consider fractional L-intersecting families of *subspaces* of an n-dimensional vector space over a finite field, instead of *subsets* of $[n]$. In particular, they show that the maximum size of a bisection-closed family of subspaces is at most $O([n]_q \log_2 n)$, where $[n]_q$ is the q-analog of the integer n. Furthermore, they exhibit examples of bisection-closed families of size at least $\Omega([n]_q)$, so the logarithmic factor that separates the upper and lower bounds persists even in the q-analog of the set version of bisection-closed families. One is strongly led to believe that removing the logarithmic factor in the set case, i.e. resolving Problem 6.6, can lead to corresponding improvements in the bounds in the q-analog case, too.

In Chapter 8, we discuss some steps towards settling this problem in the affirmative. In order to describe our results, we note that to each $M \in \mathcal{M}_n^{(f)}(\mathbf{a})$ there corresponds a tournament on the vertex set $[n]$ in the following natural manner: for $i < j$ we direct the edge ij as $i \to j$ if $M(i,j) = f(a_i, a_j)$, and the edge is directed in the reverse direction if $M(i,j) = f(a_j, a_i)$. Conversely, for a tournament T on $[n]$, we can associate the matrix $M_T^{(f)}(\mathbf{a}) \in \mathcal{M}_n^{(f)}(\mathbf{a})$ in exactly the same way, namely, for $i < j$, set $M_T^{(f)}(i,j) = f(a_i, a_j)$ if $i \to j$ and $M_T^{(f)}(i,j) = f(a_j, a_i)$ otherwise. Note that this correspondence is not necessarily one-to-one, since the a_i need not be distinct.

Our first result shows that almost all the matrices in $\mathcal{M}_n^{(f)}(\mathbf{a})$ have high rank for linear functions $f(x,y) = \alpha x + \beta y$ such that $\alpha + \beta \neq 0$, and sequences $\mathbf{a}$ of nonzero elements of a field $\mathbb{F}$ of characteristic different from 2. More precisely, we show that for a *random tournament*—a tournament with the edges being directed in either direction with probability 1/2 each and independently—then *with high probability (whp)* the rank is at least $(1/2 - o(1))n$. Here, the phrase "with high probability" means that the probability that the said event occurs asymptotically tends to 1 as $n \to \infty$.

Theorem 6.8. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$ *and* $\mathbf{a}$ *is a sequence of nonzero elements of* $\mathbb{F}$. *Let* $f\colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ *be given by* $f(x,y) = \alpha x + \beta y$, *with* $\alpha, \beta \in \mathbb{F}$ *such that* $\alpha + \beta \neq 0$. *If* T *is a uniformly random tournament, then* whp $\mathrm{rank}(M_T^{(f)}(\mathbf{a})) \geq \frac{n}{2} - 3\sqrt{n \log n}$.

To give some perspective on this result vis-à-vis the existing literature on similar problems, the behavior of random symmetric matrices is an immensely active area of research and there are several papers that consider various random models (see [42, 94] and the references therein) and Theorem 6.8 may be regarded as another addition to that list, though there is a fundamental difference between our result and all the others. For one, as we have pointed out earlier, the matrix that arises from a bisection closed family has rank at most $2n/3 + O(1)$, so in that sense our result is somewhat qualitatively different from those that appear in several of those papers. It must be pointed out that the main result in [94] considers random symmetric matrices $M_n = ((\xi_{ij}))$ where ξ_{ij} are all jointly independent (for $i < j$) and also independent of ξ_{ii} (which are also independent) with the additional property that for all $i < j$ and all real x, $\mathbb{P}(\xi_{ij} = x) \leq 1 - \mu$ for some fixed constant μ, and their result shows that *whp* the spectrum is simple. This does establish (in a strong form) Theorem 6.8 over the reals, *in the special case where a_i are all pairwise distinct*. But otherwise the best bound this suggests is of the order $\Omega(\sqrt{n})$. Secondly, our result holds over all fields $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$ whereas most other results usually work specifically with $\mathbb{R}$ or $\mathbb{C}$ (though they have stronger results).

We suspect that the above result is far from best possible, and that the rank of a uniformly random matrix should be close to full with high probability. Proving this may require techniques beyond the scope of those used in this thesis.

When $f\colon \mathbb{F}\times\mathbb{F}\to\mathbb{F}$ is the projection $f(x,y)=x$, we shall omit the superscript notation (f) and simply speak of the family $\mathcal{M}_n(\mathbf{a})$, where $\mathbf{a}$ is a sequence of nonzero elements in $\mathbb{F}$. It is a natural question to ask whether specific bounds on the rank can be proven for certain classes of tournaments, and we are able to show that for matrices in $\mathcal{M}_n(\mathbf{a})$ associated to *transitive* tournaments the rank is indeed high.

Theorem 6.9. *The transitive tournament on $[n]$ with edges oriented as $1\to 2\to\cdots\to n$ is said to be in the* natural orientation, *and is denoted T_n. If T_n is the transitive tournament on $[n]$ in the natural orientation, then $\text{rank}(M_{T_n}(\mathbf{a}))\geq n-1$. Moreover, for all $n\geq 1$, $M_{T_n}(\mathbf{a})$ and $M_{T_{n+1}}(\mathbf{a})$ cannot both be singular.*

Since any transitive tournament T on $[n]$ is isomorphic to T_n, the matrix $M_T(\mathbf{a})$ is similar to $M_{T_n}(\mathbf{a})$, and thus has the same rank. Hence, Theorem 6.9 resolves a conjecture in [15]. Also note that Theorem 6.9 holds over arbitrary fields. It is instructive to compare this with a result of de Caen's [26] on the rank of tournament matrices (where the entries are only 0 and 1). In [26], among other things, it is shown that the rank of any $n\times n$ tournament matrix is at least $\frac{n-1}{2}$ over any field and at least $n-1$ over the reals. Our result is in a similar spirit, since it answers a question in a more general setup, but is also in contrast with de Caen's since we consider symmetric matrices.

The natural correspondence between tournaments on $[n]$ and the family $\mathcal{M}_n(\mathbf{a})$ motivates the following analogous problem. Define $\mathcal{M}_{n,\text{skew}}(\mathbf{a})$ to be the family of all skew-symmetric $n\times n$ matrices over $\mathbb{F}$ such that the diagonal entries are all zero, and for $1\leq i<j\leq n$ the (i,j)th entry is either a_i or a_j. Given a tournament T on $[n]$ and a sequence $\mathbf{a}$ of non-zero elements in $\mathbb{F}$, we denote by $M_{T,\text{skew}}(\mathbf{a})$ the corresponding skew-symmetric matrix in $\mathcal{M}_{n,\text{skew}}(\mathbf{a})$, and vice-versa. One may view the skew-symmetry in our scenario as a consequence of incorporating the direction of the edges of the tournament T into the data that defines the matrix $M_{T,\text{skew}}(\mathbf{a})$, *à la* (generalized) skew-adjacency matrices of tournaments.

Let us emphasize that the ranks of general matrices associated to graphs has emerged as a powerful and useful tool in several combinatorial problems in recent times—see [41, 48, 53, 55]. For instance, questions about the rank ([65]) and spectra ([28]) of the skew-adjacency matrices of directed graphs, the existence of skew-Hadamard matrices in constructing orthogonal designs ([62]), and unimodular tournaments ([21]) are widely studied in the literature. It is also worth mentioning a remarkable result in [66] which generalizes the classical matrix-tree theorem of Kirchhoff and Tutte to 3-graphs by using a specially concocted skew-symmetric matrix. All in all, the analogous problem in our case over skew-symmetric matrices merits examination, independent of the original combinatorial motivation. We now state the problem for the sake of completeness, over an arbitrary field $\mathbb{F}$:

Problem 6.10. Is there a constant $c > 0$ such that $\mathrm{rank}(M) \geq cn$ for all $M \in \mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$?

While it is not immediately clear how to prove an analogue of the random result in Theorem 6.8 for the family $\mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$ (it is not clear if such a result even holds), we are able to show that for transitive tournaments it is true that the corresponding skew-symmetric matrices have high rank.

Theorem 6.11. *If T_n is the transitive tournament on $[n]$ in the natural orientation, then* $\mathrm{rank}(M_{T_n,\mathrm{skew}}(\mathbf{a})) \geq n-1$*. Moreover, for all $n \geq 1$, $M_{T_n,\mathrm{skew}}(\mathbf{a})$ and $M_{T_{n+1},\mathrm{skew}}(\mathbf{a})$ cannot both be singular.*

As a corollary, we then have the following result:

Corollary 6.12. *If $M \in \mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$, then* $\mathrm{rank}(M) \geq \lfloor \log_2(n) \rfloor$.

Note that Theorem 6.11 and Corollary 6.12 hold over arbitrary fields, whereas the random result in Theorem 6.8 requires $\mathrm{char}(\mathbb{F}) \neq 2$.

Chapter 7

Hierarchically intersecting families

7.1 Preliminaries

In what follows, we always assume that $\mathcal{F}$ is an r-closed θ-intersecting family over with $r \geq 3$. We denote by $\mathcal{F}(i)$ the collection of all i-element sets in $\mathcal{F}$; that is, if $\mathcal{F}$ is a family over $[n]$, then $\mathcal{F}(i) \coloneqq \mathcal{F} \cap \binom{[n]}{i}$, where $\binom{X}{i}$ denotes the collection of i-element subsets of the set X.

Our first observation is that the possible sizes that could appear in any intersection of t sets ($2 \leq t \leq r$) in $\mathcal{F}$ is quite limited.

Proposition 7.1. *Let $2 \leq t \leq r$ and suppose $A_1, \dots, A_t \in \mathcal{F}$ are distinct sets with $|A_1| \leq \cdots \leq |A_t|$. Then, $|A_1 \cap \cdots \cap A_t| \in \{\theta|A_1|, \theta|A_2|\}$.*

Proof. Since $2 \leq t \leq r$, we have $\theta|A_1| \leq |A_1 \cap \cdots \cap A_t| \leq |A_1 \cap A_2| \leq \theta|A_2|$, and so $|A_1 \cap \cdots \cap A_t| \in \{\theta|A_1|, \theta|A_2|\}$. □

Next, we show that one can often define a core of a set $A \in \mathcal{F}$ with certain nice properties.

Definition 7.2. For $A \in \mathcal{F}$, define the set $\mathsf{Tor}(A)$ of θ-intersectors of A by

$$\mathsf{Tor}(A) \coloneqq \{B \in \mathcal{F} : |B| \geq |A|,\ |A \cap B| = \theta|A|\}.$$

Note the condition $|B| \geq |A|$ in the definition of $\mathsf{Tor}(A)$.

Proposition 7.3. *If $\mathsf{Tor}(A) \neq \emptyset$, then $A \cap B = A \cap B'$ for all $B, B' \in \mathsf{Tor}(A)$.*

Proof. We have $\theta|A| \leq |A \cap B \cap B'| \leq |A \cap B| = |A \cap B'| = \theta|A|$. Hence, $A \cap B \cap B' = A \cap B$ and $A \cap B \cap B' = A \cap B'$. Thus, $A \cap B = A \cap B'$. □

Definition 7.4. For $A \in \mathcal{F}$ such that $\mathsf{Tor}(A) \neq \emptyset$, define the *core* of A by

$$\mathsf{Cor}(A) := A \cap B$$

for any $B \in \mathsf{Tor}(A)$.

Proposition 7.3 shows that Definition 7.4 is well-defined. For a set $A \in \mathcal{F}$, $\mathsf{Cor}(A)$ is not defined iff $\mathsf{Tor}(A) = \emptyset$. The next two results describe when this may happen.

Proposition 7.5. *Let $|\mathcal{F}(i)| \geq 2$. Then $\mathsf{Tor}(A) \neq \emptyset$ for all $A \in \mathcal{F}(i)$.*

Proof. If $A, B \in \mathcal{F}(i)$ are two distinct sets, then $|A \cap B| = \theta|A|$, so $B \in \mathsf{Tor}(A)$. Hence, $\mathsf{Tor}(A) \neq \emptyset$. □

Corollary 7.6. *If $A \in \mathcal{F}(i)$ such that $\mathsf{Tor}(A) = \emptyset$, then $\mathcal{F}(i) = \{A\}$.*

In fact, Proposition 7.5 implies that the family $\mathcal{F}$ is a union of uniform sunflowers.

Definition 7.7. A family $\mathcal{F}$ of subsets of $[n]$ is called a *sunflower* if, for $C := \bigcap_{A \in \mathcal{F}} A$, we have $A \cap B = C$ for all distinct $A, B \in \mathcal{F}$.

Lemma 7.8. *Every nonempty $\mathcal{F}(i)$ is a sunflower.*

Proof. If $|\mathcal{F}(i)| \leq 2$, then this is trivial. Let $|\mathcal{F}(i)| \geq 3$. To show that $|\mathcal{F}(i)|$ is a sunflower, it suffices to show that $\mathsf{Cor}(A) = \mathsf{Cor}(B)$ for any two sets $A, B \in \mathcal{F}(i)$. The proof of Proposition 7.5 shows that $A \in \mathsf{Tor}(B)$ and $B \in \mathsf{Tor}(A)$ for any two sets $A, B \in \mathcal{F}(i)$. Hence, $\mathsf{Cor}(A) = A \cap B = B \cap A = \mathsf{Cor}(B)$. □

Remark.

1. Note that the set C in Definition 7.7 is usually called the core of the sunflower. In particular, if the sunflower is a singleton set $\{A\}$, then $C = A$.

   However, our definition of core is Definition 7.4. This matches with the above notion when $|\mathcal{F}(i)| \geq 2$. But, when $\mathcal{F}(i) = \{A\}$, $\mathsf{Cor}(A)$ is either undefined (if $\mathsf{Tor}(A) = \emptyset$), or a subset of A having cardinality θi (if $\mathsf{Tor}(A) \neq \emptyset$).

2. The property of being 3-closed is crucially used in the proof of Proposition 7.3. Thus, if $\mathcal{F}$ is not 3-closed, then Definition 7.4 cannot be made, and Lemma 7.8 need not hold. Indeed, Example 6.2 shows that there are 2-bisection closed families that do not satisfy this property.

We now establish some notations that we will use throughout the rest of this chapter. Let

$$S := \{i \in [n] : \mathcal{F}(i) \neq \emptyset\}, \qquad i_{\min} := \min(S),$$

$$S_{\text{nor}} \coloneqq \{i \in S : \mathsf{Tor}(A) \neq \emptyset \text{ for all } A \in \mathcal{F}(i)\}, \qquad i_{\max} \coloneqq \max(S_{\text{nor}}),$$
$$S_{\text{exc}} \coloneqq \{i \in S : \mathsf{Tor}(A) = \emptyset \text{ for some } A \in \mathcal{F}(i)\}.$$

Note that $S = S_{\text{nor}} \sqcup S_{\text{exc}}$. We say that $\mathcal{F}(i)$ is a *normal* sunflower if $i \in S_{\text{nor}}$, and we say that it is an *exceptional* sunflower if $i \in S_{\text{exc}}$. Define $\mathcal{F}_{\text{nor}} \coloneqq \bigcup_{i \in S_{\text{nor}}} \mathcal{F}(i)$ and $\mathcal{F}_{\text{exc}} \coloneqq \bigcup_{i \in S_{\text{exc}}} \mathcal{F}(i)$. Then, $\mathcal{F} = \mathcal{F}_{\text{nor}} \sqcup \mathcal{F}_{\text{exc}}$. Define $\mathsf{Pet}(A) \coloneqq A \setminus \mathsf{Cor}(A)$ for each $A \in \mathcal{F}_{\text{nor}}$. For the sake of brevity, we also define the following:

$$\mathsf{Set}(\mathcal{F}(i)) \coloneqq \bigcup_{A \in \mathcal{F}(i)} A \qquad \text{for any } i \in S,$$
$$\mathsf{Pet}(\mathcal{F}(i)) \coloneqq \bigcup_{A \in \mathcal{F}(i)} \mathsf{Pet}(A) \qquad \text{for any } i \in S_{\text{nor}},$$
$$\mathsf{Cor}(\mathcal{F}(i)) \coloneqq \mathsf{Cor}(A) \qquad \text{for any } A \in \mathcal{F}(i),\ i \in S_{\text{nor}}.$$

Furthermore, let

$$\mathcal{F}(\geq i) \coloneqq \bigcup_{j \geq i} \mathcal{F}(j) \qquad \text{and} \qquad \mathcal{F}(I) \coloneqq \bigcup_{i \in I} \mathcal{F}(i) \quad \text{for any } I \subset [n].$$

Thus, we may also speak of $\mathsf{Pet}(\mathcal{F}(\geq i))$ and $\mathsf{Set}(\mathcal{F}(\geq i))$, as well as $\mathsf{Pet}(\mathcal{F}(I))$ and $\mathsf{Set}(\mathcal{F}(I))$ for any $I \subset [n]$.

Observation 7.9. Proposition 7.5 and Corollary 7.6 show that if $\mathsf{Tor}(A) \neq \emptyset$ for some $A \in \mathcal{F}(i)$, then $i \in S_{\text{nor}}$, and if $i \in S_{\text{exc}}$, then $|\mathcal{F}(i)| = 1$.

7.1.1 The structure of $\mathcal{F}_{\text{nor}}$

The next few results describe the structure of the normal sunflowers in $\mathcal{F}$ in relation to the cores.

Observation 7.10. The proof of Lemma 7.8 shows that if $A, B \in \mathcal{F}_{\text{nor}}$ with $|A| = |B|$, then $\mathsf{Cor}(A) = \mathsf{Cor}(B)$.

Lemma 7.11. *If $A, B \in \mathcal{F}_{nor}$ with $|A| < |B|$, then $\mathsf{Cor}(A) \subsetneq \mathsf{Cor}(B)$.*

Proof. Let $A' \in \mathsf{Tor}(A)$, $B' \in \mathsf{Tor}(B)$. Consider $A \cap A' \cap B = \mathsf{Cor}(A) \cap B \subseteq \mathsf{Cor}(A)$. Since $\theta|A| \leq |A \cap A' \cap B| \leq |\mathsf{Cor}(A)| = \theta|A|$, we have $A \cap A' \cap B = \mathsf{Cor}(A)$ and $\mathsf{Cor}(A) \subseteq B$. Since $|B| \leq |B'|$, we can run the above argument with B' in place of B to show that $\mathsf{Cor}(A) \subseteq B'$. Hence, $\mathsf{Cor}(A) \subseteq B \cap B' = \mathsf{Cor}(B)$. Lastly, $\mathsf{Cor}(A) \neq \mathsf{Cor}(B)$ because $|\mathsf{Cor}(A)| = \theta|A| \neq \theta|B| = |\mathsf{Cor}(B)|$. □

Lemma 7.12. *Suppose that $i, j \in S$ such that $i < \theta j$. If $A \in \mathcal{F}(i)$ and $B \in \mathcal{F}(j)$, then $B \in \mathsf{Tor}(A)$. In particular, $i \in S_{nor}$.*

Proof. Since $|A \cap B| \leq |A| < \theta j$, we must have $|A \cap B| = \theta i$. Hence, $B \in \mathsf{Tor}(A)$. Thus, $i \in S_{\text{nor}}$ by Observation 7.9. □

Lemma 7.13. *Let $A \in \mathcal{F}_{nor}$. If there exists $B \in \mathcal{F}(i_{\max})$ such that $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) \neq \emptyset$, then $\mathsf{Cor}(B) \subseteq A$. Moreover, there is at most one set $A \in \mathcal{F}_{nor}$ for which this happens.*

Proof. Note that $|A| < i_{\max}$ by Observation 7.10. Let $C \in \mathsf{Tor}(B)$, and consider $A \cap B \cap C = A \cap \mathsf{Cor}(B) \subseteq \mathsf{Cor}(B)$. By Lemma 7.11, $\mathsf{Cor}(A) \subseteq \mathsf{Cor}(B)$, and $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) \neq \emptyset$ by assumption. Hence, $\theta|A| < |A \cap \mathsf{Cor}(B)|$, which implies that $\theta i_{\max} \leq |A \cap B \cap C| = |A \cap \mathsf{Cor}(B)| \leq |\mathsf{Cor}(B)| = \theta i_{\max}$. Thus, $\mathsf{Cor}(B) \subseteq A$.

Now, suppose that there exists $A' \in \mathcal{F}_{\text{nor}}$ distinct from A for which there exists $B' \in \mathcal{F}(i_{\max})$ such that $\mathsf{Pet}(A') \cap \mathsf{Cor}(B') \neq \emptyset$. By Lemma 7.8, $\mathsf{Cor}(B) = \mathsf{Cor}(B')$. So, $\mathsf{Cor}(B) \subseteq A \cap A'$, which implies that $|A \cap A'| \geq \theta i_{\max}$, a contradiction. □

Denote by E_{nor} the unique set $A \in \mathcal{F}_{\text{nor}}$ for which there exists $B \in \mathcal{F}(i_{\max})$ such that $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) \neq \emptyset$, whenever it exists. Define $\mathcal{F}^*_{\text{nor}} := \mathcal{F}_{\text{nor}} \setminus \{A \in \mathcal{F}_{\text{nor}} : A = E_{\text{nor}}\}$.

Corollary 7.14. *For all $A, B \in \mathcal{F}^*_{nor}$, $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) = \emptyset$.*

Proof. If $|A| > |B|$, then $\mathsf{Cor}(A) \supsetneq \mathsf{Cor}(B)$ by Lemma 7.11, so $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) = \emptyset$. If $|A| = |B|$, then this follows from Observation 7.10. Let $|A| < |B|$, and suppose $z \in \mathsf{Pet}(A) \cap \mathsf{Cor}(B)$. Then, by Lemma 7.11, $z \in \mathsf{Cor}(B')$ for any $B' \in \mathcal{F}(i_{\max})$. Hence, $\mathsf{Pet}(A) \cap \mathsf{Cor}(B') \neq \emptyset$, which implies by Lemma 7.13 that $A = E_{\text{nor}}$, a contradiction. □

Lemma 7.11 and Corollary 7.14 say that $\mathcal{F}^*_{\text{nor}}$ has the following structure: the cores of $\mathcal{F}^*_{\text{nor}}$ form an increasing chain, and any petal is disjoint from every core. In fact, these two results can be used to show that, for $\mathcal{F}^*_{\text{nor}}$, "$r$-closed" is equivalent to "s-closed" for any $r, s \geq 3$.

Proposition 7.15. *$\mathcal{F}^*_{nor}$ is s-closed θ-intersecting for all $s \geq 2$.*

Proof. It suffices to show this for all $s > r \geq 3$, and by induction it is enough to show this for $s = r + 1$. Let $A_1, \ldots, A_{r+1} \in \mathcal{F}^*_{\text{nor}}$ be any $r + 1$ distinct sets. Without loss of generality, suppose that $|A_1| \leq \cdots \leq |A_{r+1}|$.

First, suppose that $|A_i| = |A_j|$ for some $i < j$. Then, $\mathsf{Cor}(A_1) \subseteq \bigcap_{k=1}^{r+1} A_k \subseteq \mathsf{Cor}(A_i)$ by Lemma 7.11 and Observation 7.10. But, by Corollary 7.14, $\mathsf{Pet}(A_1) \cap \mathsf{Cor}(A_i) = \emptyset$. Hence, $\bigcap_{k=1}^{r+1} A_k = \mathsf{Cor}(A_1)$. Thus, $|\bigcap_{k=1}^{r+1} A_k| = \theta|A_1|$. So, we are done in this case.

Next, suppose that $|A_i| < |A_j|$ for all $i < j$. Consider $U = A_1 \cap \cdots \cap A_r$ and $V = A_1 \cap \cdots \cap A_{r-1} \cap A_{r+1}$. By Proposition 7.1, we know that $|U|, |V| \in \{\theta|A_1|, \theta|A_2|\}$. Also, $|U \cap V| \leq \min\{|U|, |V|\}$. Note that $U \cap V = A_1 \cap \cdots \cap A_{r+1}$.

By Lemma 7.11, $\mathsf{Cor}(A_1) \subseteq U \cap V$. So, if $|U| = \theta|A_1|$ or $|V| = \theta|A_1|$, then $\theta|A_1| \leq |U \cap V| \leq \theta|A_1|$, and we are done in this case. So, assume that $|U| = \theta|A_2| = |V|$. Consider $U \subseteq A_1 \cap A_2$. Since $\theta|A_2| = |U| \leq |A_1 \cap A_2| \leq \theta|A_2|$, we have $U = A_1 \cap A_2$. Similarly, $V \subseteq A_1 \cap A_2$ and $\theta|A_2| = |V| \leq |A_1 \cap A_2| \leq \theta|A_2|$, so $V = A_1 \cap A_2$. Hence, $U \cap V = A_1 \cap A_2$, and $|U \cap V| = |A_1 \cap A_2| = \theta|A_2|$, so we are done. □

The final result of this section provides a linear upper bound on the size of $\mathcal{F}$ when $\mathcal{F} = \mathcal{F}^*_{\text{nor}}$ and $\mathsf{Tor}(A) = \{B \in \mathcal{F} : |B| \geq |A|\}$ for every $A \in \mathcal{F}$. Also, the proof technique will be used later on in the proof of Theorem 6.4 in Section 7.2.

Lemma 7.16. *Suppose that for all $A, B \in \mathcal{F}^*_{nor}$ such that $|A| < |B|$, we have $B \in \mathsf{Tor}(A)$. Then, $|\mathcal{F}^*_{nor}| \leq \lfloor \frac{n-a}{b-a} \rfloor$.*

Proof. For simplicity of notation, assume that $\mathcal{F} = \mathcal{F}^*_{\text{nor}}$. Suppose that $S = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$. Let $C := \mathsf{Cor}(\mathcal{F}(i_k))$. Define $Y_j := \mathsf{Set}(\mathcal{F}(i_j)) \setminus C$ for each $1 \leq j \leq k$. By Lemma 7.11 and Corollary 7.14, $Y_j = \mathsf{Pet}(\mathcal{F}(i_j))$ for each $1 \leq j \leq k$. Since $B \in \mathsf{Tor}(A)$ whenever $A \in \mathcal{F}(i_j)$ and $B \in \mathcal{F}(i_{j'})$ for $j < j'$, we must have $\mathsf{Pet}(A) \cap \mathsf{Pet}(B) = \emptyset$. Thus, $Y_j \cap Y_{j'} = \emptyset$ for all $j \neq j'$. Now, notice that

$$|\mathcal{F}(i_j)| = \frac{|Y_j|}{(1-\theta)i_j},$$

since the petals in $\mathcal{F}(i_j)$ are pairwise disjoint sets with each having size $(1-\theta)i_j$. Thus,

$$|\mathcal{F}| = \sum_{j=1}^{k} |\mathcal{F}(i_j)| = \sum_{j=1}^{k} \frac{|Y_j|}{(1-\theta)i_j}.$$

We also have $\sum_{j=1}^{k} |Y_j| \leq n - |C| = n - \theta i_k$. It is now easy to see that $|\mathcal{F}|$ is maximized when $|Y_j| = (1-\theta)i_j$ for $2 \leq j \leq k$, and $|Y_1|$ is the largest integer $\leq n - \theta i_k - \sum_{j=2}^{k}(1-\theta)i_j$ which is divisible by $(1-\theta)i_1$. Thus, the maximum of $|\mathcal{F}|$ taken as S varies over all subsets of $[n]$ of size k, with k varying from 1 to n, occurs when $k = 1$ and $i_1 = b$, where $\theta = a/b$ in least form, $a, b > 0$. This maximum is easily seen to be $\lfloor \frac{n-a}{b-a} \rfloor$. □

7.1.2 The structure of $\mathcal{F}_{\text{exc}}$

The next few results describe the structure of the exceptional sunflowers in $\mathcal{F}$ in relation to the cores.

Lemma 7.17. *Suppose that $S_{nor} \neq \emptyset$. Let $i \in S_{exc}$ such that $i > i_{\max}$. If $\mathcal{F}(i) = \{A\}$, then $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq A$.*

Proof. Let $B \in \mathcal{F}(i_{\max})$ and $C \in \mathsf{Tor}(B)$. Consider $A \cap B \cap C = A \cap \mathsf{Cor}(B) \subseteq \mathsf{Cor}(B)$. Since, $\theta|B| \leq |A \cap B \cap C| \leq |\mathsf{Cor}(B)| = \theta|B|$, we have $\mathsf{Cor}(B) \subseteq A$, as required. □

Lemma 7.18. *Suppose that $S_{nor} \neq \emptyset$. Let $i \in S_{exc}$ such that $i < i_{\max}$. If $\mathcal{F}(i) = \{A\}$, then, either $|A \cap \mathsf{Cor}(\mathcal{F}(i_{\max}))| = \theta i$, or $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq A$. Moreover, there is at most one $i < i_{\max}$ such that the latter case holds.*

Proof. Let $B \in \mathcal{F}(i_{\max})$ and $C \in \mathsf{Tor}(B)$. Consider $A \cap B \cap C = A \cap \mathsf{Cor}(B) \subseteq \mathsf{Cor}(B)$. If $|A \cap B \cap C| < \theta i_{\max}$, then we must have $|A \cap \mathsf{Cor}(B)| = \theta i$, which is the former case. If $|A \cap B \cap C| = \theta i_{\max}$, then $A \cap B \cap C = \mathsf{Cor}(B)$, since $|\mathsf{Cor}(B)| = \theta i_{\max}$. Hence, $\mathsf{Cor}(B) \subseteq A$, which is the latter case. Lastly, suppose for the sake of contradiction that there exists $i' \in S_{\text{exc}}$, $i' \neq i$, such that $i' < i_{\max}$, $\mathcal{F}(i') = \{A'\}$, and $\mathsf{Cor}(B) \subseteq A'$. Then, $A \cap A' \supseteq \mathsf{Cor}(B)$, so $|A \cap A'| \geq \theta i_{\max}$, which is a contradiction. □

Denote by E_{exc} the unique set in $\mathcal{F}_{\text{exc}}$ such that $|E_{\text{exc}}| < i_{\max}$ and $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq E_{\text{exc}}$, whenever it exists.

Lemma 7.19. *Let $\theta = a/b$, $\gcd(a, b) = 1$. Let $A \in \mathcal{F}$ such that $b \nmid |A|$. Then, $A \in \mathcal{F}_{exc}$, and there is at most one such set A in $\mathcal{F}$. Moreover, if $S_{nor} \neq \emptyset$, then $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq A$.*

Proof. For any $A_1 \in \mathcal{F}$ distinct from A, we must have $|A \cap A_1| = \theta|A_1|$, since $\theta|A|$ is not an integer. So, $\mathsf{Tor}(A) = \emptyset$, which implies that $A \in \mathcal{F}_{\text{exc}}$. If there were another such set A', then $|A \cap A'|$ can be neither $\theta|A|$ nor $\theta|A'|$, which is a contradiction.

Let $S_{\text{nor}} \neq \emptyset$, $B \in \mathcal{F}(i_{\max})$, and $C \in \mathsf{Tor}(B)$. Consider $A \cap B \cap C = A \cap \mathsf{Cor}(B)$. Since, $|A \cap B \cap C| \neq \theta|A|$, we have $\theta i_{\max} \leq |A \cap B \cap C| \leq |\mathsf{Cor}(B)| = \theta i_{\max}$. Hence, $A \cap B \cap C = \mathsf{Cor}(B)$, which implies that $\mathsf{Cor}(B) \subseteq A$.

□

Denote by E_θ the unique set in $\mathcal{F}$ such that $b \nmid |E_\theta|$ (where $\theta = a/b$, $\gcd(a, b) = 1$), whenever it exists. Define $\mathcal{F}^*_{\text{exc}} := \mathcal{F}_{\text{exc}} \setminus \{A \in \mathcal{F}_{\text{exc}} : A = E_{\text{exc}} \text{ or } E_\theta\}$. Define $\mathcal{F}^* := \mathcal{F}^*_{\text{nor}} \cup \mathcal{F}^*_{\text{exc}}$.

7.1.3 The structure of $\mathcal{F}^*$

Observation 7.20. If $\theta = a/b$, $\gcd(a, b) = 1$, then $|A| \equiv 0 \pmod{b}$ for all $A \in \mathcal{F}^*$.

Proposition 7.21. $|\mathcal{F}^*| \leq |\mathcal{F}| \leq |\mathcal{F}^*| + 1$.

Proof. It suffices to show that at most one of E_{nor}, E_{exc}, and E_θ can belong to the family $\mathcal{F}$. If $S_{\text{nor}} = \emptyset$, then neither E_{nor} nor E_{exc} can exist by definition. So, suppose that $S_{\text{nor}} \neq \emptyset$. Then, $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq E_{\text{nor}}$, E_{exc}, and E_θ by Lemmas 7.13, 7.18, and 7.19, respectively. Hence, the size of the intersection of any two of these sets must be at least $|\mathsf{Cor}(\mathcal{F}(i_{\max}))| = \theta i_{\max}$, which is neither $\theta|E_{\text{nor}}|$, nor $\theta|E_{\text{exc}}|$, nor $\theta|E_\theta|$, which is a contradiction. □

7.2 Proofs of Theorems 6.4 and 6.5

Assume that $\mathcal{F} = \mathcal{F}^*$. Lemma 7.12 motivates us to partition the family $\mathcal{F}$ as $\mathcal{F} = \bigsqcup_{k\geq 0} \mathcal{F}(I_k)$, where $I_k := (i_{\min}/\theta^{k-1}, i_{\min}/\theta^k]$ for $k \geq 1$, and $I_0 := \{i_{\min}\}$. Suppose that $S_{\text{nor}} \neq \emptyset$. Let $C := \mathsf{Cor}(\mathcal{F}(i_{\max}))$. Define $Y_k := \mathsf{Set}(\mathcal{F}(I_k)) \setminus C$.

Observation 7.22. If $v > u + 1$, then $Y_u \cap Y_v = \emptyset$.

Proof. It suffices to show that if $A \in \mathcal{F}(i)$ $(i \in I_u \cap S)$ and $B \in \mathcal{F}(j)$ $(j \in I_v \cap S)$, then $A \cap B \subseteq C$. It follows from the definitions of I_k, $k \geq 0$, that $i < \theta j$ for any such i and j. Thus, by Lemma 7.12, $B \in \mathsf{Tor}(A)$, so $A \cap B = \mathsf{Cor}(A)$. Hence, by Lemma 7.11, $A \cap B \subseteq C$. □

Observation 7.23.

$$\sum_{k \text{ odd}} |Y_k| \leq n - |C| \leq n - \theta i_{\min},$$
$$\sum_{k \text{ even}} |Y_k| \leq n - |C| \leq n - \theta i_{\min}.$$

Proof. This is immediate from the previous observation. □

Observation 7.24. Let $i \in I_k$. Then,

$$|\mathcal{F}(i)| \leq \frac{|Y_k|}{(1-\theta)i}.$$

Proof. Let $i \in S_{\text{nor}}$. By Lemma 7.11 and Corollary 7.14, $A \setminus C = \mathsf{Pet}(A)$ for all $A \in \mathcal{F}(i)$, so $Y_k \supseteq \mathsf{Pet}(\mathcal{F}(i))$. By Lemma 7.8, $\mathsf{Pet}(A) \cap \mathsf{Pet}(A') = \emptyset$ for all distinct $A, A' \in \mathcal{F}(i)$. Hence, $|Y_k| \geq |\mathsf{Pet}(\mathcal{F}(i))| = \sum_{A\in\mathcal{F}(i)} |\mathsf{Pet}(A)|$. Since $|\mathsf{Pet}(A)| = (1-\theta)i$ for all $A \in \mathcal{F}(i)$, we are done.

Let $i \in S_{\text{exc}}$ and $\mathcal{F}(i) = \{A\}$. First, consider the case when $i > i_{\max}$. Since $Y_k \supseteq A \setminus C$, and $C \subseteq A$ by Lemma 7.17, we have $|Y_k| \geq |A| - |C| = i - \theta i_{\max} > i - \theta i$. So, we are done. Next, consider the case when $i < i_{\max}$. Since we assume that $\mathcal{F} = \mathcal{F}^*$, we have $|A \cap C| = \theta i$ by Lemma 7.18. Hence, $|A \setminus C| = i - \theta i$. Since $Y_k \supseteq A \setminus C$, we are done. □

We also need the following result.

Lemma 7.25. *Let $\eta > 1$, and let $m \geq 1$ be an integer. Consider the sequence $(s_k)_{k\geq 1}$ given by*

$$s_k := \frac{1}{\lfloor m\eta^{k-1}\rfloor + 1} + \frac{1}{\lfloor m\eta^{k-1}\rfloor + 2} + \cdots + \frac{1}{\lfloor m\eta^{k}\rfloor}.$$

Then, $\lim_{k\to\infty} s_k = \ln(\eta)$.

When η is an integer, the sequence $(s_k)_{k\geq 1}$ is monotonically increasing to $\ln(\eta)$. *In general, $s_k < \ln(\eta) + \frac{1}{m}$ for all $k \geq 1$.*

Proof. Let H_n denote the nth harmonic number, $H_n = \sum_{i=1}^{n} 1/i$. It is well-known that $\lim_{n\to\infty}(H_n - \ln(n)) = \gamma$, the Euler–Mascheroni constant. Hence,

$$s_k = H_{\lfloor m\eta^k\rfloor} - H_{\lfloor m\eta^{k-1}\rfloor} = \ln\left(\frac{\lfloor m\eta^k\rfloor}{\lfloor m\eta^{k-1}\rfloor}\right) + \epsilon(\lfloor m\eta^k\rfloor) - \epsilon(\lfloor m\eta^{k-1}\rfloor),$$

where $\lim_{n\to\infty} \epsilon(n) = 0$. Since

$$\eta - \frac{1}{m\eta^{k-1}} < \frac{\lfloor m\eta^k\rfloor}{\lfloor m\eta^{k-1}\rfloor} < \frac{\eta}{1 - \frac{1}{m\eta^{k-1}}},$$

we have $\lim_{k\to\infty} s_k = \ln(\eta)$.

When η is an integer, the monotonicity of the sequence $(s_k)_{k\geq 1}$ is a corollary of the following more general observation, where $n \geq 1$ is any integer:

$$\sum_{i=n+1}^{\eta n} \frac{1}{i} < \sum_{i=n+1}^{\eta n} \frac{1}{i} + \left(\frac{1}{\eta n + 1} - \frac{1}{\eta n + \eta}\right) + \cdots + \left(\frac{1}{\eta n + \eta - 1} - \frac{1}{\eta n + \eta}\right) = \sum_{i=(n+1)+1}^{\eta(n+1)} \frac{1}{i}.$$

To show that $s_k < \ln(\eta) + \frac{1}{m}$ for all $k \geq 1$, observe that

$$\begin{aligned} s_k &< \int_{\lfloor m\eta^{k-1}\rfloor}^{\lfloor m\eta^k\rfloor} \frac{1}{t} dt \\ &\leq \ln(m\eta^k) - \ln(\lfloor m\eta^{k-1}\rfloor) \\ &= \ln(\eta) + \ln\left(\frac{m\eta^{k-1}}{\lfloor m\eta^{k-1}\rfloor}\right) \\ &< \ln(\eta) + \frac{1}{\lfloor m\eta^{k-1}\rfloor} \\ &\leq \ln(\eta) + \frac{1}{m}. \end{aligned}$$

□

7.2.1 Proof of Theorem 6.4

We restate the theorem below for convenience:

Theorem 6.4. *Let $\mathcal{F}$ be an r-closed θ-intersecting family over $[n]$, with $r \geq 3$. Let $\theta = a/b \in (0,1)$ with $\gcd(a,b) = 1$, $a, b > 0$.*

1. *If $a > 1$, then $|\mathcal{F}| \leq 2\left(\frac{\ln(b)-\ln(a)+1}{b-a}\right)(n-a)+1$.*
2. *If $a = 1$, then we have two cases:*
   (a) *if $b = 2$ and $\mathcal{F}$ contains a set of size 2, then $|\mathcal{F}| \leq (1+\ln(2))(n-1)+1$;*
   (b) *otherwise, $|\mathcal{F}| \leq \left(\frac{2\ln(b)}{b-1}\right)(n-1)+1$.*

Proof. We assume throughout that $\mathcal{F} = \mathcal{F}^*$, since it suffices to compute $|\mathcal{F}^*|$ by Proposition 7.21.

First, observe that if $\mathcal{F}_{\text{nor}} = \emptyset$, then only $\mathcal{F}(I_0)$ and $\mathcal{F}(I_1)$ can be nonempty by Lemma 7.12. Furthermore, each nonempty $\mathcal{F}(i)$ is a singleton set. Therefore, $|\mathcal{F}| \leq \frac{1}{b}\left(\frac{i_{\min}}{\theta} - i_{\min}\right)+1$, which is maximized when $n = \frac{i_{\min}}{\theta}$. Hence, this gives the bound $|\mathcal{F}| \leq \lfloor\frac{(1-\theta)n}{b}\rfloor + 1$, which is stronger than those in the statement of Theorem 6.4.

For the rest of the proof, suppose that $\mathcal{F}_{\text{nor}} \neq \emptyset$. Let $i_{\min} = mb$ for some $m \geq 1$ by Observation 7.20. For $k \geq 1$, we have

$$|\mathcal{F}(I_k)| = \sum_{i \in I_k \cap S} |\mathcal{F}(i)| \leq \sum_{i \in I_k \cap S} \frac{|Y_k|}{(1-\theta)i} \leq \begin{cases} \frac{|Y_k|}{b-a}\left(\ln(\theta^{-1}) + \frac{1}{m}\right), & a > 1; \\ \frac{|Y_k|}{b-1}(\ln(b)), & a = 1, \end{cases}$$

from Observations 7.20 and 7.24, as well as Lemma 7.25. For $k = 0$, we have

$$|\mathcal{F}(I_0)| = |\mathcal{F}(i_{\min})| \leq \frac{|Y_0|}{(1-\theta)mb} \leq \begin{cases} \frac{|Y_0|}{b-a}\left(\ln(\theta^{-1}) + \frac{1}{m}\right), & a > 1; \\ \frac{|Y_0|}{b-1}\left(\frac{1}{m}\right), & a = 1, \end{cases}$$

from Observation 7.24 and Lemma 7.25. Since

$$|\mathcal{F}| = \sum_{k \geq 0} |\mathcal{F}(I_k)| = \sum_{k \text{ odd}} |\mathcal{F}(I_k)| + \sum_{k \text{ even}} |\mathcal{F}(I_k)|,$$

we get the bound

$$|\mathcal{F}| \leq 2\left(\frac{\ln(b) - \ln(a) + 1}{b-a}\right)(n - |C|) \tag{7.1}$$

when $a > 1$ by applying Observation 7.23.

When $a = 1$, we need to compare the term $1/m$ appearing in the bound for $\mathcal{F}(I_0)$ with the term $\ln(b)$ appearing in the bound for $\mathcal{F}(I_k)$ for k even: since $1/m > \ln(b)$ if and only if $m = 1$ and $b = 2$, and this happens if and only if $\theta = 1/2$ and $i_{\min} = 2$, we get

$$\sum_{k \text{ odd}} |\mathcal{F}(I_k)| \leq \frac{\ln(b)}{b-1} \sum_{k \text{ odd}} |Y_k|, \qquad \sum_{k \text{ even}} |\mathcal{F}(I_k)| \leq \begin{cases} \dfrac{1}{2-1} \displaystyle\sum_{k \text{ even}} |Y_k|, & \theta = 1/2,\ i_{\min} = 2; \\ \dfrac{\ln(b)}{b-1} \displaystyle\sum_{k \text{ even}} |Y_k|, & \text{otherwise.} \end{cases}$$

Thus, by Observation 7.23,

$$|\mathcal{F}| \leq \begin{cases} (1 + \ln(2))(n - |C|), & \theta = 1/2 \text{ and } i_{\min} = 2; \\ \left(\dfrac{2\ln(b)}{b-1}\right)(n - |C|), & \text{otherwise.} \end{cases} \tag{7.2}$$

The result now follows immediately from (7.1) and (7.2). □

7.2.2 Proof of Theorem 6.5

We restate the theorem for convenience:

Theorem 6.5. *Let $\mathcal{F}$ be an r-bisection closed family over $[n]$, with $r \geq 3$. Then,*

$$|\mathcal{F}| \leq \lfloor \tfrac{3n}{2} \rfloor - 2 \tag{$*$}$$

for all $n \geq 2$. Moreover:

1. *(Tightness) For each $n \geq 2$, there exists an r-bisection closed family $\mathcal{F}_{\max}$ over $[n]$ which attains the above bound.*
2. *(Uniqueness) For any family $\mathcal{F}$ over $[n]$ that attains the above bound, there is a permutation σ of $[n]$ such that $\mathcal{F}_{\max} = \sigma(\mathcal{F}) := \{\sigma(A) : A \in \mathcal{F}\}$, where $\sigma(E) := \{\sigma(a) : a \in E\}$ for any set $E \in \mathcal{P}([n])$.*
3. *(Stability) There exists an absolute constant $C > 0$ such that the following holds. If $|\mathcal{F}| \geq (\frac{3}{2} - \epsilon)n$ for some $0 < \epsilon < 0.1$, then for some permutation σ of $[n]$,*

$$|\sigma(\mathcal{F}) \setminus \mathcal{F}_{\max}| < C\epsilon n.$$

We begin with an outline of the proof of Theorem 6.5 before presenting the details. Since the theorem is easily verified for $n = 2, 3$, we may assume that $n \geq 4$. We also assume that $\mathcal{F} = \mathcal{F}^*$ by Proposition 7.21. First, we show that the upper bound on $|\mathcal{F}|$ holds when $S = S_{\text{nor}} = \{2, 4\}$. Second, we show that if $S_{\text{nor}} \not\supseteq \{2, 4\}$, then $\mathcal{F}$ cannot be an extremal family. Finally, we show that if $S_{\text{nor}} \supsetneq \{2,4\}$, then we can get a family that is strictly larger than $\mathcal{F}$ by removing all the sets of sizes greater than 4 and adding new sets of sizes 2 and 4. The uniqueness and stability are then easily verified, thus completing the proof.

Proof. Example 6.1 constructs an r-bisection closed family $\mathcal{F}$ such that $|\mathcal{F}| = \lfloor \frac{3n}{2} \rfloor - 2$ for any $n \geq 2$, so the bound ($*$), which we shall establish below, is in fact tight. Assume that $n \geq 4$. We will restrict our attention to the class of families $\mathcal{F}$ for which $\mathcal{F} = \mathcal{F}^*$.

Claim 1. If $S = S_{\text{nor}} = \{2,4\}$, then $|\mathcal{F}| \leq \lfloor \frac{3n}{2} \rfloor - 3$.

Proof. Let $\mathsf{Cor}(\mathcal{F}(2)) = \{a_1\}$ and $\mathsf{Cor}(\mathcal{F}(4)) = \{a_1, a_2\}$. By Corollary 7.14 it follows that $|\mathcal{F}(4)| \leq \lfloor \frac{n-2}{2} \rfloor$ and $|\mathcal{F}(2)| \leq n-2$ so $|\mathcal{F}| = |\mathcal{F}(2)| + |\mathcal{F}(4)| \leq \lfloor \frac{3n}{2} \rfloor - 3$. □

Claim 2. If $S_{\text{nor}} \not\supseteq \{2,4\}$, then $\mathcal{F}$ is not an extremal family.

Proof. Suppose for the sake of contradiction that $\mathcal{F}$ is extremal. Let $C := \mathsf{Cor}(\mathcal{F}(i_{\max}))$.

If $S = \{2,4\}$ but $S_{\text{exc}} \neq \emptyset$, then clearly there cannot be more than n sets in the family $\mathcal{F}$, contradicting its extremality. So, assume that $S \neq \{2,4\}$.

Theorem 6.4 already shows that $|\mathcal{F}| < \lfloor \frac{3n}{2} \rfloor - 3$ for a bisection closed family unless $i_{\min} = 2$. So, suppose that $2 \in S$.

If $2 \in S_{\text{exc}}$, then there cannot be any $i \in S$ such that $i > 4$ by Lemma 7.12. So, $S = \{2\} = S_{\text{exc}}$, but this implies that $|\mathcal{F}| = 1$, contradicting the extremality of $\mathcal{F}$. Hence, $2 \in S_{\text{nor}}$.

Next, if $4 \notin S$, then by Lemma 7.12, $A \cap B = \mathsf{Cor}(A)$ for all $A \in \mathcal{F}(2)$, $B \in \mathcal{F}(\geq 6)$. If $n = 4$, then $\mathcal{F}(\geq 6) = \emptyset$, so we must have $S = \{2\}$. However, this contradicts the extremality of $\mathcal{F}$, as we have seen earlier, so assume that $n \geq 6$. Let $m_1 = |\mathsf{Pet}(\mathcal{F}(2))|$ and $m_2 = |\mathsf{Set}(\mathcal{F}(\geq 6))|$. Then, $m_1 + m_2 \leq n$, and $|\mathcal{F}| \leq m_1 + \lfloor 2\ln(2)(m_2 - |C|) \rfloor \leq 1 + \lfloor 2\ln(2)(n-4) \rfloor$ by (7.2). This is less than $\lfloor \frac{3n}{2} \rfloor - 3$, which contradicts the extremality of $\mathcal{F}$. So, $4 \in S$.

Lastly, if $4 \in S_{\text{exc}}$, then $S \subset \{2,4,6,8\}$ by Lemma 7.12. Suppose that $\mathcal{F}(8) \neq \emptyset$. Then, if $\mathcal{F}(4) = \{A\}$, we must have $|A \cap B| = \frac{1}{2}|B| = |A|$ for any $B \in \mathcal{F}(8)$. Hence, $A \subset B$ for all $B \in \mathcal{F}(8)$. So, if $8 \in S_{\text{nor}}$, then $A = \mathsf{Cor}(B)$, implying that $A = E_{\text{exc}}$. This contradicts that $\mathcal{F} = \mathcal{F}^*$, so $8 \notin S_{\text{nor}}$. But then $\mathcal{F}(4)$ and $\mathcal{F}(8)$ together contain at most two sets, and it is easy to see by a similar argument as in the previous case that $|\mathcal{F}|$ is strictly less than $\lfloor \frac{3n}{2} \rfloor - 3$, which contradicts the extremality of $\mathcal{F}$. So, $4 \in S_{\text{nor}}$. □

Now, additionally assume that $\mathcal{F}$ is an extremal family, so that $S_{\text{nor}} \supseteq \{2,4\}$.

Claim 3. If there exists $a \in \mathsf{Pet}(\mathcal{F}(2)) \cap A$ for some $A \in \mathcal{F}(\geq 4)$, then $A \in \mathcal{F}(4)$ and $a \in \mathsf{Pet}(A)$.

Proof. This follows from Observation 7.22 and Corollary 7.14. □

Claim 4. $|\mathsf{Pet}(\mathcal{F}(2)) \setminus \mathsf{Pet}(\mathcal{F}(4))| \leq 1$.

Proof. Suppose for the sake of contradiction that $a_1, a_2 \in \mathsf{Pet}(\mathcal{F}(2)) \setminus \mathsf{Pet}(\mathcal{F}(4))$ such that $a_1 \neq a_2$. Define $B' := \mathsf{Cor}(\mathcal{F}(4)) \cup \{a_1, a_2\}$ and $\mathcal{F}' := \mathcal{F} \cup \{B'\}$. By Observation 7.22, $a_1, a_2 \notin \mathsf{Set}(\mathcal{F}'(\geq 6))$, so $\mathcal{F}'$ is r-bisection closed. But, $|\mathcal{F}'| > |\mathcal{F}|$, which contradicts the maximality of $\mathcal{F}$. □

Claim 5. Let $B \in \mathcal{F}(4)$ and $\mathsf{Pet}(B) = \{a, b\}$. Then:

1. $|\{a, b\} \cap \mathsf{Pet}(\mathcal{F}(2))| \in \{0, 2\}$, or
2. $|\{a, b\} \cap \mathsf{Pet}(\mathcal{F}(2))| = 1$, and if $b \in \mathsf{Pet}(\mathcal{F}(2))$, then there is a unique set $A \in \mathcal{F}(\geq 6)$ such that $a \in A$. Moreover, $A \in \mathcal{F}(6)$.

Proof. Suppose that $b \in \mathsf{Pet}(\mathcal{F}(2))$ and $a \notin \mathsf{Pet}(\mathcal{F}(2))$. If $a \notin B'$ for any $B' \in \mathcal{F}$ distinct from B, then we contradict the extremality of $\mathcal{F}$ as follows: the family $\mathcal{F}' := \mathcal{F} \cup \{A'\}$, where $A' := \mathsf{Cor}(\mathcal{F}(2)) \cup \{a\}$, is r-bisection closed and satisfies $|\mathcal{F}'| > |\mathcal{F}|$.

So, there is a set $A \in \mathcal{F}$ distinct from B for which $a \in A$. In particular, $A \in \mathcal{F}(\geq 6)$. Note that $\mathsf{Cor}(B) \cup \{a\} \subseteq A$, so $|A \cap B| \geq 3 > \frac{1}{2}|B|$. Thus, $|A \cap B| = \frac{1}{2}|A|$. So, if $A \in \mathcal{F}(\geq 8)$, then in fact $A \in \mathcal{F}(8)$ and $B \subseteq A$. But this implies that $b \in A$, which contradicts Claim 3. Thus, $A \in \mathcal{F}(6)$.

Lastly, if $\mathcal{F}(6)$ is a singleton, then A is clearly unique, and if there are at least two sets in $\mathcal{F}(6)$, then $a \notin A'$ for any $A' \in \mathcal{F}(6)$ distinct from A because $\mathcal{F}(6)$ is a sunflower and $a \in \mathsf{Pet}(A)$. □

Claim 6. If Claim 5(1) holds for all $B \in \mathcal{F}(4)$, then $S_{\text{nor}} = \{2, 4\}$.

Proof. Partition the family $\mathcal{F}$ into two disjoint nonempty subfamilies as follows: let $\mathcal{G}_1$ be the subfamily consisting of the sets in $\mathcal{F}(2)$ as well as those sets B in $\mathcal{F}(4)$ such that $\mathsf{Pet}(B) \cap \mathsf{Pet}(\mathcal{F}(2)) \neq \emptyset$, and let $\mathcal{G}_2$ be the subfamily of $\mathcal{F}$ containing the remaining sets. Let $m_1 = |\mathsf{Pet}(\mathcal{G}_1)|$ and $m_2 = |\mathsf{Set}(\mathcal{G}_2)|$. By Claim 5(1), $\mathsf{Pet}(A) \cap B = \emptyset$ for all $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. So, $m_1 + m_2 \leq n$. Also, $i_{\min}(\mathcal{G}_2) \geq 4$, and $|C| \geq 3$ since $S \supsetneq \{2, 4\}$. Observe that $|\mathcal{G}_1| = \lfloor \frac{3m_1}{2} \rfloor$ and $|\mathcal{G}_2| \leq \lfloor 2\ln(2)(m_2 - 3) \rfloor$ by (7.2). But then $|\mathcal{F}| = |\mathcal{G}_1| + |\mathcal{G}_2| \leq \lfloor \frac{3m_1}{2} \rfloor + \lfloor 2\ln(2)(m_2 - 3) \rfloor < \lfloor \frac{3n}{2} \rfloor - 3$ since $m_2 \geq 6$, which contradicts the extremality of $\mathcal{F}$. □

Claim 7. If $|\mathcal{F}(2)|$ is maximum, then Claim 5(1) holds for all $B \in \mathcal{F}(4)$.

Proof. Suppose for the sake of contradiction that $|\mathcal{F}(2)|$ is maximum and Claim 5(2) holds for some $B \in \mathcal{F}(4)$, so there is a unique set $A \in \mathcal{F}(\geq 6)$ such that $a \in A$, and in fact $A \in \mathcal{F}(6)$. Now, consider the family $\mathcal{F}'' := (\mathcal{F} \setminus \{A\}) \cup \{A''\}$, where $A'' := \mathsf{Cor}(\mathcal{F}(2)) \cup \{a\}$. Again, the property of being r-bisection closed is preserved, and $|\mathcal{F}''| = |\mathcal{F}|$, but $|\mathcal{F}''(2)| > |\mathcal{F}(2)|$, which is a contradiction. □

Thus, if $\mathcal{F}$ is any extremal r-bisection closed family (not necessarily one for which $\mathcal{F} = \mathcal{F}^*$), then the bound $(*)$ holds, since it holds in particular for those extremal families $\mathcal{F}$ for which $|\mathcal{F}(2)|$ is maximum. We now prove tightness and uniqueness for such families:

Lemma 7.26. *Let $\mathcal{F}$ be an extremal r-bisection closed family over $[n]$ for which $S_{nor} = \{2, 4\}$. Then, there is a permutation σ of $[n]$ such that $\sigma(\mathcal{F}) = \mathcal{F}_{\max}$, where $\mathcal{F}_{\max}$ is the family constructed in Example 6.1. In particular, if $\mathcal{F}$ is an extremal family for which $|\mathcal{F}(2)|$ is maximum, then $\sigma(\mathcal{F}) = \mathcal{F}_{\max}$ for some permutation σ of $[n]$.*

Proof. As noted before, the family constructed in Example 6.1 is tight for the upper bound $(*)$, and we shall call that family $\mathcal{F}_{\max}$. Note that $\mathcal{F}_{\max} = \mathcal{F}_{\max}(2) \sqcup \mathcal{F}_{\max}(4)$ (so $S_{\text{nor}} = \{2, 4\}$), and that $\mathcal{F}_{\max}$ is r-bisection closed for any $r \geq 2$. This proves the tightness of $(*)$ when $S_{\text{nor}} = \{2, 4\}$. Also note that $E_{\text{nor}} = \{1, 2\}$ belongs to the family $\mathcal{F}_{\max}$.

For the uniqueness, let $\mathcal{F}$ be any extremal family for which $S_{\text{nor}} = \{2, 4\}$. Claim 1 shows that we must have $|\mathcal{F}^*| = \lfloor \frac{3n}{2} \rfloor - 3$, and in particular $|\mathcal{F}^*(2)| = n - 2$ and $|\mathcal{F}^*(4)| = \lfloor \frac{n-2}{2} \rfloor$. That is, assuming $\mathsf{Cor}(\mathcal{F}^*(2)) = \{a_1\}$ and $\mathsf{Cor}(\mathcal{F}^*(4)) = \{a_1, a_2\}$, the sets in $\mathcal{F}^*(2)$ are precisely all those obtained by taking the union of $\{a_1\}$ with singleton sets $\{b\}$ such that $b \neq a_1, a_2$, and the sets in $\mathcal{F}^*(4)$ are precisely all those obtained by taking the union of $\{a_1, a_2\}$ with two-element sets $\{b_1, b_2\}$ that are pairwise disjoint from each other as well as from $\{a_1, a_2\}$. Since $\mathcal{F}^*$ is an intersecting family, it is r-bisection closed, too. A moment's reflection shows that this family $\mathcal{F}^*$ can be obtained simply by applying an appropriate permutation of $[n]$ to $\mathcal{F}^*_{\max}$.

To complete the analysis, observe that $\mathcal{G} := \mathcal{F}^* \cup \{\mathsf{Cor}(\mathcal{F}(4))\}$ is also r-bisection closed, and the permutation of $[n]$ that mapped $\mathcal{F}^*$ to $\mathcal{F}^*_{\max}$ also maps $\mathcal{G}$ to $\mathcal{F}_{\max}$. Clearly, $E_{\text{nor}}(\mathcal{G}) = \mathsf{Cor}(\mathcal{F}(4))$. To show that $\mathcal{G} = \mathcal{F}$, we verify that neither E_{exc} nor E_θ can belong to $\mathcal{F}$. Suppose $E_{\text{exc}} \in \mathcal{F}$. Then $\mathsf{Cor}(\mathcal{F}(4)) \subsetneq E_{\text{exc}}$. But, if $\{a\} \neq \mathsf{Cor}(\mathcal{F}(2))$, then $a \in \mathsf{Pet}(A)$ for some $A \in \mathcal{F}(2)$. In particular, we must have $A \cap E_{\text{exc}} = A$ which forces $E_{\text{exc}} \in \mathcal{F}(4)$, but this is a contradiction. The same argument also shows that $E_\theta \notin \mathcal{F}$. Lastly, since $E_{\text{nor}}(\mathcal{F})$ is a set which contains $\mathsf{Cor}(\mathcal{F}(4))$, either $E_{\text{nor}}(\mathcal{F}) = \mathsf{Cor}(\mathcal{F}(4))$ $(= E_{\text{nor}}(\mathcal{G}))$, or $E_{\text{nor}}(\mathcal{F}) \in \mathcal{F}(4)$. In the latter case, $E_{\text{nor}}(\mathcal{F}) = \mathsf{Cor}(\mathcal{F}(4)) \cup \{b, b'\}$ for some $b, b' \in [n]$, but then its intersection with the unique set in $\mathcal{F}^*$ containing b will intersect $E_{\text{nor}}(\mathcal{F})$ in a set of size three, namely $\mathsf{Cor}(\mathcal{F}(4)) \cup \{b\}$, and this contradicts that $\mathcal{F}$ is bisection-closed. Thus, $\mathcal{F} = \mathcal{G}$, and this completes the proof of uniqueness of the extremal family.

Finally, note that by Claims 6 and 7, if $\mathcal{F}$ is an extremal family for which $|\mathcal{F}(2)|$ is maximum, then $S_{\text{nor}} = \{2, 4\}$. So, the lemma also holds for extremal families over $[n]$ having the maximum number of sets of size 2, which must be $n-1$ as witnessed by $\mathcal{F}_{\max}$. □

To complete the proof of tightness and uniqueness, it remains to show that there is no extremal family $\mathcal{F}$ for which $|\mathcal{F}(2)| < n-1$.

Proposition 7.27. *There is no extremal family $\mathcal{F}$ over $[n]$ for which $|\mathcal{F}(2)| < n-1$.*

Proof. Suppose for the sake of contradiction that $\mathcal{F}$ is an extremal family over $[n]$ for which $|\mathcal{F}(2)| < n-1$. Then, $S_{\text{nor}} \supsetneq \{2, 4\}$ by Lemma 7.26 and Claim 3. Also, Claim 5(2) holds for some $B \in \mathcal{F}^*(4)$ by Claim 6 and Lemma 7.26.

Now, let $\mathcal{F}_0 := \mathcal{F}^*$. For $n \in \mathbb{N}$, suppose that the extremal r-bisection closed family $\mathcal{F}_n$ has been defined, such that $\mathcal{F}_n = \mathcal{F}_n^*$ and there is a set $B_n \in \mathcal{F}_n(4)$ for which Claim 5(2) holds. Then, we define $\mathcal{F}_{n+1}$ as follows. Let $\mathsf{Pet}(B_n) = \{a_n, b_n\}$ with $b_n \in \mathsf{Pet}(\mathcal{F}_n(2))$. Let $A_n \in \mathcal{F}_n(6)$ be the unique set in $\mathcal{F}_n(\geq 6)$ such that $a_n \in A_n$. Then, define $\mathcal{F}_{n+1} := (\mathcal{F}_n \setminus \{A_n\}) \cup \{A'_n\}$, where $A'_n := \mathsf{Cor}(\mathcal{F}_n(2)) \cup \{a_n\}$. Note that $\mathcal{F}_{n+1}$ is also an r-bisection closed family that is extremal, since $|\mathcal{F}_n| = |\mathcal{F}_{n+1}|$. Since the set A'_n is not E_{nor}, E_{exc}, or E_θ, we also have $\mathcal{F}_{n+1} = \mathcal{F}^*_{n+1}$.

Applying this procedure inductively by starting with $\mathcal{F}_0 := \mathcal{F}$, for some $N \in \mathbb{N}$ we get an extremal family $\mathcal{F}' = \mathcal{F}_N$ such that $\mathcal{F}' = \mathcal{F}'^*$ and Claim 5(1) holds for all $B' \in \mathcal{F}'(4)$. Hence, by Claim 6, $\mathcal{F}'$ has only two normal sunflowers, namely $\mathcal{F}'(2)$ and $\mathcal{F}'(4)$. Since the only sets from $\mathcal{F}_0 = \mathcal{F}^*$ that were thrown out in the construction of $\mathcal{F}'$ were those of size 6, $\mathcal{F}^*$ has only three normal sunflowers, namely $\mathcal{F}^*(2)$, $\mathcal{F}^*(4)$, and $\mathcal{F}^*(6)$. Now, let $B \in \mathcal{F}^*(6)$, and let $\mathsf{Pet}(B) = \{a, b, c\}$. Define $\mathcal{G} = (\mathcal{F}^* \setminus \{B\}) \cup \{D_a, D_b, D_c\}$, where $D_i := \mathsf{Cor}(\mathcal{F}(2)) \cup \{i\}$, for $i \in \{a, b, c\}$. Then, $\mathcal{G}$ is an r-bisection closed family for which $|\mathcal{G}| \geq |\mathcal{F}| + 1$, contradicting the extremality of $\mathcal{F}$. □

This proves that the family $\mathcal{F}_{\max}$ over $[n]$ of Example 6.1 is the unique extremal r-bisection closed family (up to permutations of $[n]$).

Lastly, we prove the stability as follows. Theorem 6.4 and the proof of the upper bound $(*)$ show that $|\mathcal{F}| < 2\ln(2)(n-1) + 1$ for any r-bisection closed family $\mathcal{F}$ that is not extremal. Since $\frac{3}{2} - 2\ln(2) \approx 0.11$, the claim follows. □

7.3 Concluding remarks

We ignore all floors and ceilings here for simplicity.

- While Theorem 6.5 considers the maximum size among all possible r-bisection closed families, it is possible to consider a more constrained problem:

  Problem 7.28. For an integer $k \geq 2$, determine the maximum size of an r-bisection closed family $\mathcal{F}$ with $i_{\min}(\mathcal{F}^*) \geq k$.

  Theorem 6.4 establishes a linear upper bound, and it is not hard to construct a heirarchically bisection closed family of size at least $(2n - k - 4)\left(\frac{1}{k} + \frac{1}{k+2} + \frac{1}{k+4}\right)$ when $k \equiv 0 \pmod 4$. Our methods in this paper suggest that all the possible set sizes must lie in the range $[k, 2k]$ for an optimal family. There could be more than three distinct set sizes in an optimal family, though it seems rather unlikely that sets of all possible sizes in this range can be attained. Settling this question fully may require other new ideas.

- While Theorem 6.5 gives a tight result for $\theta = 1/2$, the bound in Theorem 6.4 in the general case is far from best possible. Again, one can mimic the construction for $\mathcal{F}_{\max}$ to get r-closed θ-intersecting families of size $(n - 2a)\left(\frac{1}{b-a} + \frac{1}{2(b-a)}\right)$ if $\theta = \frac{a}{b}$, but this is not best possible in general. If $\theta = 1/b$ for b odd, then one can get a heirarchically closed θ-intersecting family $\mathcal{F}$ of size $(n-3)\left(\frac{1}{b-1} + \frac{1}{2(b-1)} + \frac{1}{3(b-1)}\right)$. If $\theta = 1/b$ for $b \equiv 0 \pmod 4$, then in general one can get a heirarchically closed θ-intersecting family $\mathcal{F}$ of size $(n-4)\left(\frac{1}{b-1} + \frac{1}{2(b-1)} + \frac{1}{4(b-1)}\right)$. Similar constructions can be made in general when $a \neq 1$. The methods in this paper suggest that the best bound ought to be attained when $i_{\min}(\mathcal{F}^*)$ is as small as possible, i.e. $i_{\min}(\mathcal{F}^*) = b$ when $\theta = a/b$ in least form, but a complete answer seems beyond the scope of the methods in this paper.

  Problem 7.29. For a fraction $\theta = a/b \in (0, 1)$, determine the maximum size of an r-closed θ-intersecting family $\mathcal{F}$.

- The following general question naturally arises from the above two problems, and we make the explicit statement for the sake of completeness:

  Problem 7.30. For a fraction $\theta = a/b \in (0, 1)$ and an integer $k \geq b$, determine the maximum size of an r-closed θ-intersecting family $\mathcal{F}$ with $i_{\min}(\mathcal{F}^*) = k$.

- Another interesting question arises as an artifact of our proof ideas. If $\mathcal{F} = \mathcal{F}_{\text{exc}}$ then the proof of Theorem 6.4 also shows that $|\mathcal{F}| \leq \left(\frac{1-\theta}{b}\right) n + 2$. But, it appears that this bound is far from best possible, and we believe that in this case $|\mathcal{F}| = O(\sqrt{n})$. Since the notion of an exceptional family seems a bit contrived, a more natural question is the following:

Question 7.31. *Suppose $\mathcal{F} = \{A_1, \ldots, A_m\}$ is an r-closed θ-intersecting family with $|A_i| < |A_j|$ whenever $i < j$. Is $|\mathcal{F}| \leq O(\sqrt{n})$?*

One indication that this bound is the correct order comes from the situation when $|A_i \cap A_j| = \theta|A_i|$ whenever $i < j$. This setup is similar to that in Lemma 7.16, but under the additional constraint that there is at most one set of any fixed size. Indeed, in this case, a straightforward inductive argument shows that $|\bigcup_{i=1}^{k} A_i| \geq k^2$, and that gives the bound stated. But in the general case, the methods developed in this paper seem to fall short of being able to settle this conjecture in the affirmative. The following weaker version of the above question could prove to be more amenable to investigation:

Question 7.32. *Suppose $\mathcal{F} = \{A_1, \ldots, A_m\}$ is an r-closed θ-intersecting family with $|A_i| < |A_j|$ whenever $i < j$. Is $|\mathcal{F}| \leq o(n)$?*

Chapter 8

Ranks of matrices arising from tournaments

8.1 Proof of Theorem 6.8

We need a version of McDiarmid's inequality for concentration bounds on product measure spaces, and we use the one stated in [69, Lemma 1.2]:

Theorem 8.1 (Independent Bounded Differences Inequality)**.** *Let $X_1, \dots, X_n$ be independent random variables, with X_k taking values in a set Ω_k for each k. Suppose that the measurable function $g\colon \prod_k \Omega_k \to \mathbb{R}$ satisfies, for each k,*

$$|g(\mathbf{x}) - g(\mathbf{x}')| \le c_k$$

whenever the vectors $\mathbf{x}$ and $\mathbf{x}'$ differ only in the kth coordinate. Let Y be the random variable $g(X_1, \dots, X_n)$. Then, for any $t > 0$,

$$\mathbb{P}(|Y - \mathbf{E}(Y)| > t) \;\le\; 2\exp\bigl(-2t^2/\textstyle\sum_k c_k^2\bigr).$$

We restate Theorem 6.8 for convenience:

Theorem 6.8. *Suppose* $\operatorname{char}(\mathbb{F}) \neq 2$ *and* $\mathbf{a}$ *is a sequence of nonzero elements of* $\mathbb{F}$*. Let* $f\colon \mathbb{F}\times\mathbb{F} \to \mathbb{F}$ *be given by* $f(x,y) = \alpha x + \beta y$*, with* $\alpha, \beta \in \mathbb{F}$ *such that* $\alpha + \beta \neq 0$*. If T is a uniformly random tournament, then* whp $\operatorname{rank}(M_T^{(f)}(\mathbf{a})) \ge \frac{n}{2} - 3\sqrt{n\log n}$.

Proof. The notation $[m, n]$ denotes the set of integers i such that $m \le i \le n$. Let $\Omega_k = \{0,1\}^{[k+1,n]}$ for $k = 1, \dots, n-1$. View each vector $\mathbf{x}_k = (x_k^{(k+1)}, \dots, x_k^{(n)}) \in \Omega_k$ as a win-loss record for player k against the players $k+1, \dots, n$ in that order. Thus, any $(n-1)$-tuple $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})$, where $\mathbf{x}_k \in \Omega_k$ for each k, determines a unique tournament T

on $[n]$, and each tournament on $[n]$ arises from some point in $\prod_k \Omega_k$. Define $g\colon \prod_k \Omega_k \to \mathbb{R}$ by $g(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \mathrm{rank}(M_T^{(f)}(\mathbf{a}))$, where T is the tournament uniquely determined by $\mathbf{x}_1, \dots, \mathbf{x}_{n-1}$.

We begin with a couple of simple observations.

1. Changing the orientation of any one edge ij to get a tournament T' changes the rank by at most 2, since the matrix $M_{T'}^{(f)}(\mathbf{a})$ is obtained from $M_T^{(f)}(\mathbf{a})$ by adding a matrix of rank at most 2 (we have $M_{T'}^{(f)}(\mathbf{a}) = M_T^{(f)}(\mathbf{a}) + B$ for a matrix B comprising of zeros everywhere except at the (i,j) and (j,i) positions). In fact, for any fixed i, flipping the orientations of any subcollection of the edges ij, for $j > i$, changes the rank by at most 2, since this again corresponds to adding a matrix of rank at most 2 to $M_T^{(f)}(\mathbf{a})$.

2. For a tournament T, let T_R denote the *reverse* tournament, i.e., if $i \to j$ in T then $j \to i$ in T_R. Then $M_T^{(f)}(\mathbf{a}) + M_{T_R}^{(f)}(\mathbf{a}) = M$ where $M(i,i) = 0$ and $M(i,j) = (\alpha+\beta)(a_i + a_j)$. In particular, $M = (\alpha+\beta)(DJ + JD - 2D)$ where D is the diagonal matrix $\mathrm{diag}(a_1, \dots, a_n)$ and J represents, as before, the all-ones matrix. In particular, since $\alpha+\beta \neq 0$, $a_i \neq 0$ and $\mathrm{char}(\mathbb{F}) \neq 2$, it follows that $\mathrm{rank}(M) \geq n-2$. Consequently, at least one of $\mathrm{rank}(M_T^{(f)}(\mathbf{a}))$ and $\mathrm{rank}(M_{T_R}^{(f)}(\mathbf{a}))$ is at least $n/2 - 1$.

By the first observation, $|g(\mathbf{x}) - g(\mathbf{x}')| \leq 2$ whenever $\mathbf{x}$ and $\mathbf{x}'$ differ only in the kth coordinate. Now, let T be a uniformly random tournament on $[n]$, i.e., one for which the orientation of each edge ij is chosen by a fair coin toss. Then, T gives rise to random variables X_k taking values in Ω_k for each $1 \leq k \leq n-1$, and $X_1, \dots, X_{n-1}$ are independent. Define $Y := g(X_1, \dots, X_{n-1})$. By the second observation, $\mathbf{E}(Y) \geq \frac{n}{2} - 1$.

Now, we apply McDiarmid's inequality to get:

$$\mathbb{P}\left(Y < \frac{n}{2} - 1 - 2\sqrt{n \log n}\right) \leq 2e^{-\frac{8n\log n}{4(n-1)}} < \frac{2}{n^2},$$

which proves the result.

□

8.2 Further remarks

- Our error probability of $O(n^{-2})$ is easily improved to $O\big(e^{-n^{1/3}}\big)$ if we take $t = n^{2/3}$ for instance in the proof of Theorem 6.8.
- As indicated in the remarks in the introduction, it must be possible to improve upon the results obtained here when we restrict ourselves to the fields $\mathbb{R}$ or $\mathbb{C}$, or if the elements a_i themselves satisfy other constraints. For instance, when the sequence $\mathbf{a}$

is the constant sequence $(a, a, \dots)$, $\operatorname{rank}(M) \geq n-1$ for all $M \in \mathcal{M}_n(\mathbf{a})$ and over any field. This also shows that if $\mathbb{F}$ is any finite field, then $\operatorname{rank}(M) \geq \frac{n}{|\mathbb{F}|-1} - 1$ for all $M \in \mathcal{M}_n(\mathbf{a})$ for *any* sequence $\mathbf{a}$ in $\mathbb{F}$. On the other hand, it is not even clear whether these results extend to the case when $\operatorname{char}(\mathbb{F}) = 2$, or when infinitely many of the a_i are distinct.

- In the statement of Theorem 6.8, we have considered the class of matrices $\mathcal{M}_n^{(f)}(\mathbf{a})$ where f is the linear function $f(x, y) = \alpha x + \beta y$. However, our method of proof suffices to prove a linear lower bound (*whp*) for matrix families arising from a wider class of functions f, specifically those of finite rank. More precisely, let $\operatorname{char}(\mathbb{F}) \neq 2$, and suppose that $f\colon \mathbb{F}^2 \to \mathbb{F}$ is a function such that $f(x,x) \neq 0$ for all x. Suppose that there exist functions $g_i, h_i \colon \mathbb{F} \to \mathbb{F}$, $1 \leq i \leq k$, such that $f(x,y) = \sum_{i=1}^k g_i(x)h_i(y)$. For $\mathbf{X} \in \mathbb{F}^n$, define $G_i(\mathbf{X}) := (g_1(\mathbf{X}), \dots, g_k(\mathbf{X}))$ and $H_i(\mathbf{X}) := (h_1(\mathbf{X}), \dots, h_k(\mathbf{X}))$ for all $1 \leq i \leq k$. Then, for any tournament T on $[n]$ and any sequence $\mathbf{a}$ in $\mathbb{F}^*$, we have $M_T^{(f)}(\mathbf{a}) + M_{T_R}^{(f)}(\mathbf{a}) = \sum_{i=1}^k \big(G_i(\mathbf{a}_n)^T H_i(\mathbf{a}_n) + H_i(\mathbf{a}_n)^T G_i(\mathbf{a}_n)\big) - 2\operatorname{diag}\big(f(a_1,a_1), \dots, f(a_n,a_n)\big)$. The RHS is the sum of a diagonal matrix—of full rank—and $2k$ matrices of rank one. Hence, at least one of $M_T^{(f)}$ or $M_{T_R}^{(f)}$ has rank at least $(n-2k)/2$. Hence, the proof of Theorem 6.8 above shows that, even in this case, we have *whp* $\operatorname{rank}\big(M_T^{(f)}(\mathbf{a})\big) \geq \frac{n}{2} - o(n)$ for a uniformly random tournament T on $[n]$.

  It is worth noting that the condition $f(x,x) \neq 0$ for all x is crucial. For example, if $f(x,y) = (x-y)^2$, then $\operatorname{rank}(M_T^{(f)}(\mathbf{a})) \leq 3$ for any tournament T and any sequence $\mathbf{a}$, since $M_T^{(f)}(\mathbf{a})$ is the sum of three rank one matrices, namely those induced by the functions x^2, $-2xy$ and y^2. This example is discussed in [48, Theorem 2.4] for the case when $\mathbb{F} = \mathbb{Q}$, and $\mathbf{a}_n = (1, 2, \dots, n)$. Thus, we ask the following question: for what functions $f\colon \mathbb{F}^2 \to \mathbb{F}$ does there exist a constant $c > 0$ such that $\operatorname{rank}(M) \geq cn$ for all $M \in \mathcal{M}_n^{(f)}(\mathbf{a})$?

- For a given sequence $\mathbf{a}$ in $\mathbb{F}$ and a matrix $M = M_T(\mathbf{a}_n) \in \mathcal{M}_n(\mathbf{a})$ arising from a self-dual tournament T, the matrix $M_{T_R}(\mathbf{a}_n)$ arising from the reverse tournament T_R can also be viewed as $M_T(\sigma\mathbf{a}_n)$ for a permutation σ of $\mathbf{a}_n = (a_1, \dots, a_n)$. We have shown that at least one of M_T and M_{T_R} has rank at least $n/2 - 1$ for any tournament T on $[n]$. An interesting question is whether, for a fixed tournament T, the matrices $M_T(\sigma\mathbf{a}_n)$ have the same rank for all permutations σ of $\mathbf{a}_n$. A positive answer to this question will tell us, in particular, that matrices arising from self-dual tournaments (such as Paley tournaments) all have high rank.

8.3 Matrices arising from transitive tournaments

Lemma 8.2. *Let* $\mathbf{a}$ *be a sequence of non-zero elements in the field* $\mathbb{F}$*. For each* $n \geq 1$*, let* T_n *be the transitive tournament on* $[n]$ *with the natural orientation.*

1. *For all* $n \geq 2$*,* $\det(M_{T_n}(\mathbf{a}))$ *satisfies the recurrence relation*

$$\det(M_{T_n}(\mathbf{a})) = -a_{n-1}^2 \det(M_{T_{n-2}}(\mathbf{a})) - 2a_{n-1} \det(M_{T_{n-1}}(\mathbf{a})), \tag{8.1}$$

where $M_{T_1}(\mathbf{a})$ *is the* 1×1 *zero matrix, and* $M_{T_0}(\mathbf{a})$ *is taken to be the empty matrix, so that* $\det(M_{T_1}(\mathbf{a})) = 0$ *and* $\det(M_{T_0}(\mathbf{a})) = 1$*.*

2. *For all* $n \geq 2$*,* $\det(M_{T_n,\mathrm{skew}}(\mathbf{a}))$ *satisfies the recurrence relation*

$$\det(M_{T_n,\mathrm{skew}}(\mathbf{a})) = a_{n-1}^2 \det(M_{T_{n-2},\mathrm{skew}}(\mathbf{a})), \tag{8.2}$$

where $M_{T_1,\mathrm{skew}}(\mathbf{a})$ *is the* 1×1 *zero matrix, and* $M_{T_0,\mathrm{skew}}(\mathbf{a})$ *is taken to be the empty matrix, so that* $\det(M_{T_1,\mathrm{skew}}(\mathbf{a})) = 0$ *and* $\det(M_{T_0,\mathrm{skew}}(\mathbf{a})) = 1$*.*

Proof. We shall prove part 1 below; the proof of part 2 will proceed along similar lines.

Since $\det(M_{T_2}(\mathbf{a})) = -a_1^2 = -a_1^2 \det(M_{T_0}(\mathbf{a})) - 2a_1 \det(M_{T_1}(\mathbf{a}))$, recurrence (8.1) is verified for $n = 2$. Fix $n \geq 3$, and view a_{n-1} as a variable. Then, $\det(M_{T_n}(\mathbf{a}))$ is a formal polynomial in a_{n-1} of degree at most 2, since a_{n-1} occurs only in the $(n-1, n)$ and $(n, n-1)$ positions in $M_{T_n}(\mathbf{a})$. Observe that for a field $\mathbb{F}$ of any characteristic, the constant term of a polynomial over $\mathbb{F}$ can be found by applying the evaluation map that sends the indeterminate to $0 \in \mathbb{F}$; similarly, the coefficient of the linear term of a polynomial over $\mathbb{F}$ can be found by first computing its formal derivative and then applying the same evaluation map.

In our case, the constant term of the polynomial $\det(M_{T_n}(\mathbf{a}))$ must be zero, since by plugging in $a_{n-1} = 0$ the last two columns of $M_{T_n}(\mathbf{a})$ become identical. Write $\det(M_{T_n}(\mathbf{a})) = \alpha a_{n-1}^2 + \beta a_{n-1}$, and expand the determinant as

$$\det(M_{T_n}(\mathbf{a})) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} [M_{T_n}(\mathbf{a})]_{i,\sigma(i)}, \tag{8.3}$$

where $\mathfrak{S}_n$ is the set of permutations of $[n]$. The terms containing a_{n-1}^2 in the RHS are only obtained from those $\sigma \in \mathfrak{S}_n$ such that $\sigma(n-1) = n$, $\sigma(n) = n-1$. Viewing $\mathfrak{S}_{n-2}$ as the set of those permuations in $\mathfrak{S}_n$ that fix $n-1$ and n, and denoting the transposition that swaps $n-1$ and n by $(n-1,n)$, we see that

$$\alpha = \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma(n)=n-1,\\ \sigma(n-1)=n}} \mathrm{sgn}(\sigma)\prod_{i=1}^{n-2}[M_{T_n}(\mathbf{a})]_{i,\sigma(i)} = \sum_{\tau\in\mathfrak{S}_{n-2}} \mathrm{sgn}(\tau\circ(n-1,n))\prod_{i=1}^{n-2}[M_{T_n}(\mathbf{a})]_{i,\tau(i)} = -\det(M_{T_{n-2}}(\mathbf{a})).$$

The coefficient of a_{n-1} can be found by computing $\left.\frac{d(\det M_{T_n}(\mathbf{a}))}{da_{n-1}}\right|_{a_{n-1}=0}$, where $\frac{d}{da_{n-1}}$ is the formal derivative operator, applied on the polynomial $\det(M_{T_n}(\mathbf{a}))$. Writing $M_{T_n}(\mathbf{a})$ as

$$M_{T_n}(\mathbf{a}) = \begin{pmatrix} M_{T_{n-2}}(\mathbf{a}) & \mathbf{a}_{n-2}^T & \mathbf{a}_{n-2}^T \\ \mathbf{a}_{n-2} & 0 & a_{n-1} \\ \mathbf{a}_{n-2} & a_{n-1} & 0 \end{pmatrix},$$

we get

$$\frac{d(\det M_{T_n}(\mathbf{a}))}{da_{n-1}} = \begin{vmatrix} M_{T_{n-2}}(\mathbf{a}) & \mathbf{a}_{n-2}^T & \mathbf{a}_{n-2}^T \\ \mathbf{0}_{n-2} & 0 & 1 \\ \mathbf{a}_{n-2} & a_{n-1} & 0 \end{vmatrix} + \begin{vmatrix} M_{T_{n-2}}(\mathbf{a}) & \mathbf{a}_{n-2}^T & \mathbf{a}_{n-2}^T \\ \mathbf{a}_{n-2} & 0 & a_{n-1} \\ \mathbf{0}_{n-2} & 1 & 0 \end{vmatrix},$$

where $\mathbf{0}_{n-2} = (0,\ldots,0) \in \mathbb{F}^{n-2}$. Substitute $a_{n-1} = 0$ and expand the first determinant along the $(n-1)$th row and the second determinant along the nth row to get that $\beta = -2\det(M_{T_{n-1}}(\mathbf{a}))$. This proves the recurrence (8.1) for all $n \geq 3$.

The proof of part 2 proceeds analogously. Since $M_{T_n,\text{skew}}(\mathbf{a})$ is skew-symmetric, the expression that was found for α in the proof of part 1 is precisely the coefficient of $-a_{n-1}^2$, rather than of a_{n-1}^2. After taking the formal derivative of $\det(M_{T_n,\text{skew}}(\mathbf{a}))$ with respect to a_{n-1} and then setting $a_{n-1} = 0$, the two determinants on the RHS cancel each other: expanding along the same rows chosen in the proof of part 1 easily shows that the two expressions only differ by a sign. This suffices to prove the recurrence (8.2). □

Theorem 8.3. *Let $\mathbb{F}$ be a field and $\mathbf{a}$ be a sequence of non-zero elements in $\mathbb{F}$. Let T_n be the transitive tournament on $[n]$ in the natural orientation. Then, for all $n \geq 1$,*

1. $\mathrm{rank}(M_{T_n}(\mathbf{a}))$, $\mathrm{rank}(M_{T_n,\text{skew}}(\mathbf{a})) \geq n-1$;
2. $M_{T_n}(\mathbf{a})$ *(resp. $M_{T_n,\text{skew}}(\mathbf{a})$) and $M_{T_{n+1}}(\mathbf{a})$ (resp. $M_{T_{n+1},\text{skew}}(\mathbf{a})$) cannot both be singular.*

Proof. We shall prove the above theorem for the symmetric matrices $M_{T_n}(\mathbf{a})$. The proof is by induction. Note that $\det(M_{T_1}(\mathbf{a})) = 0$ and $\det(M_{T_2}(\mathbf{a})) = -a_1^2 \neq 0$. Hence, $\mathrm{rank}(M_{T_1}(\mathbf{a})) \geq 0$ and $\mathrm{rank}(M_{T_2}(\mathbf{a})) \geq 1$; moreover, $M_{T_1}(\mathbf{a})$ is singular and $M_{T_2}(\mathbf{a})$ is non-singular. This verifies the base case.

For the induction hypothesis, suppose that $\text{rank}(M_{T_n}(\mathbf{a})) \geq n-1$, and moreover that if $M_{T_{n-1}}(\mathbf{a})$ is singular then $M_{T_n}(\mathbf{a})$ is non-singular, for some $n \geq 2$. Consider $M_{T_{n+1}}(\mathbf{a})$. Since $M_{T_n}(\mathbf{a})$ is a submatrix of $M_{T_{n+1}}(\mathbf{a})$, if $M_{T_n}(\mathbf{a})$ is non-singular, then $\text{rank}(M_{T_{n+1}}(\mathbf{a})) \geq n$, as required. So, suppose that $M_{T_n}(\mathbf{a})$ is singular. By the recurrence (8.1),

$$\det(M_{T_{n+1}}(\mathbf{a})) = -a_n^2 \det(M_{T_{n-1}}(\mathbf{a})) - 2a_n \det(M_{T_n}(\mathbf{a})) = -a_n^2 \det(M_{T_{n-1}}(\mathbf{a})).$$

Now, by the induction hypothesis, $\det(M_{T_{n-1}}(\mathbf{a})) \neq 0$, so $\det(M_{T_{n+1}}(\mathbf{a})) \neq 0$. Hence, $\text{rank}(M_{T_{n+1}}(\mathbf{a})) = n+1 \geq n$, and $M_{T_{n+1}}(\mathbf{a})$ is non-singular, as required.

This completes the proof for the symmetric matrices $M_{T_n}(\mathbf{a})$. The proof for the skew-symmetric matrices $M_{T_n,\text{skew}}(\mathbf{a})$ follows analogously by applying the recurrence (8.2) in place of the recurrence (8.1). □

We conclude this section with a few remarks concerning the case $\mathbb{F} = \mathbb{C}$. Here, one can define the families $\mathcal{M}_n^{\mathbb{C}}(\mathbf{a})$ and $\mathcal{M}_{n,\text{skew}}^{\mathbb{C}}(\mathbf{a})$ of Hermitian and skew-Hermitian matrices, respectively, corresponding to tournaments on $[n]$ in the analogous fashion. This naturally raises questions analogous to Problems 6.6 and 6.10, respectively. We note that one can suitably modify the proof of Lemma 8.2 to a more combinatorial flavour to show the following:

Theorem 8.4. *Let $\mathbf{a}$ be a sequence of non-zero elements in the field $\mathbb{F}$. For each $n \geq 1$, let T_n be the transitive tournament on $[n]$ with the natural orientation. For all $n \geq 2$, we have the following recurrences:*

$$\det(M_{T_n}^{\mathbb{C}}(\mathbf{a})) = -|a_{n-1}|^2 \det(M_{T_{n-2}}^{\mathbb{C}}(\mathbf{a})) - 2\Re(a_{n-1} \det(M_{T_{n-1}}^{\mathbb{C}}(\mathbf{a}))),$$
$$\det(M_{T_n,\text{skew}}^{\mathbb{C}}(\mathbf{a})) = |a_{n-1}|^2 \det(M_{T_{n-2},\text{skew}}^{\mathbb{C}}(\mathbf{a})) + 2i\Im(a_{n-1} \det(M_{T_{n-1},\text{skew}}^{\mathbb{C}}(\mathbf{a}))),$$

where $M_{T_n}^{\mathbb{C}}(\mathbf{a}) \in \mathcal{M}_n^{\mathbb{C}}(\mathbf{a})$ and $M_{T_n,\text{skew}}^{\mathbb{C}}(\mathbf{a}) \in \mathcal{M}_{n,\text{skew}}^{\mathbb{C}}(\mathbf{a})$ for all n.

As before, $M_{T_1}^{\mathbb{C}}(\mathbf{a})$ (and $M_{T_1,\text{skew}}^{\mathbb{C}}(\mathbf{a})$) is the 1×1 zero matrix, and $M_{T_0}^{\mathbb{C}}(\mathbf{a})$ (and $M_{T_0,\text{skew}}^{\mathbb{C}}(\mathbf{a})$) is taken to be the empty matrix. Hence, $\det(M_{T_1}^{\mathbb{C}}(\mathbf{a})) = 0 = \det(M_{T_1,\text{skew}}^{\mathbb{C}}(\mathbf{a}))$ and $\det(M_{T_0}^{\mathbb{C}}(\mathbf{a})) = 1 = \det(M_{T_0,\text{skew}}^{\mathbb{C}}(\mathbf{a}))$.

Proof. Since our method of proof for Lemma 8.2 requires us to view $\det(M_{T_n}(\mathbf{a}))$ as a polynomial with respect to a_{n-1}, the same line of reasoning cannot be applied *mutatis mutandis* to Theorem 8.4 since a priori $\det(M_{T_n}^{\mathbb{C}}(\mathbf{a}))$ has expressions involving both a_{n-1} and $\overline{a_{n-1}}$. Instead, we offer a suitably modified combinatorial proof, which will also apply equally well to Lemma 8.2.

In the formula (8.3) for the expansion of the determinant applied to $M^{\mathbb{C}}_{T_n}(\mathbf{a})$, we split the sum in the RHS over the sets A_1, A_2, A_3, and A_4, where

$$\begin{aligned} A_1 &:= \{\sigma \in \mathfrak{S}_n : \sigma(n-1) = n, \sigma(n) = n-1\}, \\ A_2 &:= \{\sigma \in \mathfrak{S}_n : \sigma(n-1) = n, \sigma(n) \neq n-1\}, \\ A_3 &:= \{\sigma \in \mathfrak{S}_n : \sigma(n-1) \neq n, \sigma(n) = n-1\}, \\ A_4 &:= \{\sigma \in \mathfrak{S}_n : \sigma(n-1) \neq n, \sigma(n) \neq n-1\}. \end{aligned}$$

Note that $\mathfrak{S}_n$ is the disjoint union of the A_i's. Furthermore, we can define sets $B_i \subset A_i$ consisting of those permutations that are also fixed-point free. Since all the diagonal entries of $M^{\mathbb{C}}_{T_n}(\mathbf{a})$ are zero, the sum in the RHS over the set A_i equals the sum over the set B_i for each $1 \leq i \leq 4$.

Now, the sum over A_1 is easily seen to equal $-\det(M^{\mathbb{C}}_{T_{n-2}}(\mathbf{a})) \cdot a_{n-1} \cdot \overline{a_{n-1}}$, by following the same line of argument as in the proof of Lemma 8.2. The sum over B_4 (and hence over A_4) equals zero for the following reason: the map $\sigma \mapsto \sigma \circ (n-1, n)$, where $(n-1, n)$ denotes the transposition that swaps $n-1$ and n, is a well-defined sign-reversing involution on B_4 that is invariant on the expression $\prod_{i=1}^{n}[M^{\mathbb{C}}_{T_n}(\mathbf{a})]_{i,\sigma(i)}$. Lastly, the map $\sigma \mapsto \sigma^{-1}$ is a well-defined sign-preserving bijection from A_2 to A_3 such that $\prod_{i=1}^{n}[M^{\mathbb{C}}_{T_n}(\mathbf{a})]_{i,\sigma(i)} = \overline{\prod_{i=1}^{n}[M^{\mathbb{C}}_{T_n}(\mathbf{a})]_{i,\sigma^{-1}(i)}}$, so the sum over A_2 and A_3 is together equal to twice the real part of the sum over A_2. The latter is easily seen to equal $-a_{n-1} \cdot \det(M^{\mathbb{C}}_{T_{n-1}}(\mathbf{a}))$, by first swapping the last two rows and then comparing the expression for $a_{n-1} \cdot \det(M^{\mathbb{C}}_{T_{n-1}}(\mathbf{a}))$ with the sum over A_2.

This completes the proof of the first recurrence. The proof of the second recurrence goes similarly, taking into account the appropriate sign changes. This is reflected in the term $|a_{n-1}|^2$ appearing as the coefficient attached to $\det(M^{\mathbb{C}}_{T_{n-2},\text{skew}}(\mathbf{a}))$ instead of $-|a_{n-1}|^2$, as well as in the imaginary part of the sum over A_2 appearing in place of the real part. □

8.4 Further remarks

- The recurrence (8.1) in Lemma 8.2 allows one to deduce slightly more about the (non)-singularity of the matrices $M_{T_n}(\mathbf{a})$. Indeed, if $\operatorname{char}(\mathbb{F}) \neq 2$ and, for some $n \geq 1$, $M_{T_n}(\mathbf{a})$ is singular, then both $M_{T_{n+1}}(\mathbf{a})$ and $M_{T_{n+2}}(\mathbf{a})$ must be non-singular, since the recurrence

$$\det(M_{T_{n+2}}(\mathbf{a})) = -a_{n+1}^2 \det(M_{T_n}(\mathbf{a})) - 2a_{n+1} \det(M_{T_{n+1}}(\mathbf{a}))$$

  implies that if $M_{T_{n+2}}(\mathbf{a})$ is also singular, then $a_{n+1} = 0$, a contradiction.

Furthermore, Theorem 6.9 is best possible: if $\mathrm{char}(\mathbb{F}) \neq 2$, then the sequence $\mathbf{a} = (a_1, a_2, a_3, \dots)$ whose entries are defined recursively by

$$a_n := \begin{cases} 1, & n \equiv 1, 2 \pmod 3; \\ -\dfrac{2\det(M_{T_{n-1}}(\mathbf{a}))}{\det(M_{T_{n-2}}(\mathbf{a}))}, & n \equiv 0 \pmod 3), \end{cases}$$

satisfies the condition $\mathrm{rank}(M_{T_n}(\mathbf{a})) = n - 1$ iff $n \equiv 0 \pmod 3$.

- Similarly, the recurrence (8.2) shows that $M_{T_{2n},\mathrm{skew}}(\mathbf{a})$ is always non-singular, and $M_{T_{2n+1},\mathrm{skew}}(\mathbf{a})$ is always singular; in particular,

$$\det(M_{T_{2n},\mathrm{skew}}(\mathbf{a})) = \prod_{i=1}^{n} a_{2i-1}^2$$

for all n. Thus, Theorem 6.11 is also best possible.

Of course, it is well-known that the determinant of an $n \times n$ skew-symmetric matrix vanishes when n is odd, and is the square of a polynomial in the entries of the matrix (called the Pfaffian) when n is even, and the above formula verifies this. However, we note that the recurrence (8.1) does not appear to be amenable to a simple closed-form formula in a similar fashion. Also observe that when $\mathrm{char}(\mathbb{F}) = 2$ the two families $\mathcal{M}_n(\mathbf{a})$ and $\mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$ are identical. So, if $\mathrm{char}(\mathbb{F}) = 2$, then $\mathrm{rank}(M_{T_n}) = n - 1$ iff $n \equiv 0 \pmod 2$.

- For a sequence $\mathbf{a}$ of non-zero elements in an ordered field $\mathbb{F}$, such as $\mathbb{F} = \mathbb{R}$, consider the symmetric $n \times n$ matrices $M(\overline{\mathbf{a}})$ and $M(\underline{\mathbf{a}})$ defined by

$$M(\overline{\mathbf{a}})_{i,j} = \max\{a_i, a_j\} \quad \text{and} \quad M(\underline{\mathbf{a}})_{i,j} = \min\{a_i, a_j\}$$

for all $i < j$. We also consider, similarly, the skew-symmetric versions of these $n \times n$ matrices, denoted $M(\overline{\mathbf{a}}, \mathrm{skew})$ and $M(\underline{\mathbf{a}}, \mathrm{skew})$.

Theorem 8.3 implies that these matrices all have rank at least $n-1$. Such max- and min-type matrices are natural to consider in various contexts (for instance, see [9]), so it is interesting to note that they are all of full rank or nearly so.

- Furthermore, since any tournament on $[n]$ contains a transitive subtournament of size at least $\lfloor \log_2(n) \rfloor + 1$ (see [91]), any skew-symmetric matrix $M \in \mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$ has rank at least $\lfloor \log_2(n) \rfloor$. This proves Corollary 6.12.

- Additionally, one may consider such matrices having constant (non-zero) diagonal as well. If the diagonal entry, say d, differs from all the off-diagonal entries, then the rank is at least $n-2$, since we get a matrix of the above form with zero diagonal by subtracting dJ, where J is the all-ones matrix, which has rank 1.

- The analogues of Theorem 8.3 and Corollary 6.12 hold for the families $\mathcal{M}_n^{\mathbb{C}}(\mathbf{a})$ and $\mathcal{M}_{n,\text{skew}}^{\mathbb{C}}(\mathbf{a})$, with their proofs going through in a similar fashion. Furthermore, the second remark in this section applies equally well to these matrices, too.

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