

Some recent work on fractional intersecting families

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Based on joint works with

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Fractional θ -intersecting families

Definition (Balachandran–Mathew–Mishra 2019)

Let $0 < \theta < 1$ be a rational. A family \mathcal{F} of subsets of $[n]$ is called **θ -intersecting** if for all $A, B \in \mathcal{F}$, $A \neq B$, we have

$$|A \cap B| = \theta|A| \quad \text{or} \quad |A \cap B| = \theta|B|.$$

"B θ -intersects A"

"A θ -intersects B"

Note: when $\theta = 1/2$, we also say that "A bisects B", and so on.

Question.

What is the maximum size of a fractional θ -intersecting family over $[n]$?

Example (Sunflower family)

Let $\mathcal{F}_s := \{12, 13, \dots, 1n, 1234, 1256, \dots, 12(n-1)n\}$.

Then, \mathcal{F}_s is $\frac{1}{2}$ -intersecting, and $|\mathcal{F}_s| = \frac{3n}{2} - 2$.

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Example (Hadamard family)

Let H be an $m \times m$ Hadamard matrix in normal form, and let J be the $m \times m$ all-ones matrix. Let A_1, \dots, A_{3m} be the rows of

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix},$$

viewed as the $\{\pm 1\}$ -incidence vectors of subsets of $[2m]$.

Then, $\mathcal{F}_H := \{A_i : i \in [3m] \setminus \{1, m+1\}\}$ is a $\frac{1}{2}$ -intersecting family.

Writing $2m = n$, we have $|\mathcal{F}_H| = 3n/2 - 2$.

Question.

Are these families extremal?

Even a linear upper bound is not known!

Theorem (Balachandran–Mathew–Mishra 2019)

Let $\theta \in (0, 1) \cap \mathbb{Q}$. If \mathcal{F} is a θ -intersecting family over $[n]$, then

$$|\mathcal{F}| \leq O_\theta(n \log(n)).$$

Conjecture (Balachandran–Mathew–Mishra 2019)

Let $\theta \in (0, 1) \cap \mathbb{Q}$. There is a constant $c > 0$ such that any θ -intersecting family over $[n]$ has size at most cn .

A closer look at the two examples

These two examples are at the extreme ends of a tower of *hierarchically* $\frac{1}{2}$ -intersecting families.

- ▶ In the sunflower family: for any $r \geq 2$ and any sets $A_1, \dots, A_r \in \mathcal{F}_s$ we also have

$$|A_1 \cap \dots \cap A_r| \in \left\{ \frac{1}{2}|A_1|, \dots, \frac{1}{2}|A_r| \right\}.$$

- ▶ In the Hadamard family: there is no $r > 2$ for which this property is satisfied.

Hierarchically r -closed fractional θ -intersecting families

Definition

Let $r \geq 2$ and $\theta \in (0, 1) \cap \mathbb{Q}$. A family \mathcal{F} of subsets of $[n]$ is called **hierarchically r -closed θ -intersecting** if, for each $2 \leq t \leq r$ and any t distinct sets A_1, \dots, A_t in \mathcal{F} we have $|\bigcap_{i=1}^t A_i| \in \{\theta|A_i| : 1 \leq i \leq t\}$.

- ▶ If a θ -intersecting family is r -closed, then it is also s -closed for all $2 \leq s \leq r$.
- ▶ If $r = 2$, then this reduces to the earlier notion of fractional θ -intersecting families.

Question.

What is the maximum size of a hierarchically r -closed θ -intersecting family over $[n]$, when $r \geq 3$?

Hierarchically closed families

\mathcal{F}_s is a prototypical hierarchically closed family.

In fact, if \mathcal{F} is any hierarchically closed family, then

- ▶ each nonempty subfamily $\mathcal{F}(i)$ of i -sized sets is a sunflower;

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- ▶ the cores of the nontrivial sunflowers form an increasing sequence;
- ▶ all petals and cores are disjoint from each other (with at most one exception: look at the set 12 in $\mathcal{F}_s = \{12, 13, \dots, 1n, 1234, 1256, \dots, 12(n-1)n\}$).

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Call a θ -intersecting family with these properties to be a **bouquet** of sunflowers.

Question.

Does any bouquet over $[n]$ have size linear in n ?

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Answer.

Yes!

Hierarchically closed families are linear in size

Theorem

(Balachandran–Bhattacharya–Kher–Mathew–S. 2023)

There is a constant $c_\theta \leq \frac{3}{2}$ such that, if \mathcal{F} is an r -closed θ -intersecting family over $[n]$ with $r \geq 3$, then $|\mathcal{F}| \leq c_\theta n$.

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When $\theta = 1/2$,

$$|\mathcal{F}| \leq \lfloor \frac{3n}{2} \rfloor - 2 \quad (*)$$

for all $n \geq 2$.

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When $\theta = 1/2$,

$$|\mathcal{F}| \leq \lfloor \frac{3n}{2} \rfloor - 2 \quad (*)$$

for all $n \geq 2$. Moreover:

1. (Tightness) The bound in (*) is attained by \mathcal{F}_s .
2. (Uniqueness) For any family \mathcal{F} over $[n]$ that attains the bound in (*), there is a permutation σ of $[n]$ such that $\mathcal{F}_s = \sigma(\mathcal{F}) := \{\sigma(A) : A \in \mathcal{F}\}$.
3. (Stability) There exists an absolute constant $C > 0$ such that the following holds. If $|\mathcal{F}| \geq (\frac{3}{2} - \epsilon)n$ for some $0 < \epsilon < 0.1$, then for some permutation σ of $[n]$, $|\sigma(\mathcal{F}) \setminus \mathcal{F}_s| < C\epsilon n$.

Another look at the two examples

Recall the sunflower and Hadamard families, \mathcal{F}_s and \mathcal{F}_H .

- ▶ The sunflower family: has sets of sizes 2 and 4.
- ▶ The Hadamard family: has sets of sizes $n/2$ and $n/4$.

Question.

If the sets in a θ -intersecting family \mathcal{F} are not very large, then is the size of \mathcal{F} linear in n ?

Bounded θ -intersecting families ...

- ▶ Say that a family \mathcal{F} is **w-bounded** if all the sets in \mathcal{F} have size at most w .
- ▶ We now know that *bouquets* are linear in size.
- ▶ So, if we can throw away a small number of sets from a w -bounded family \mathcal{F} to get a bouquet, then we will obtain a linear bound.
- ▶ Here w cannot be large, because of the Hadamard family \mathcal{F}_H .

Bounded θ -intersecting families ...

Proposition (Balachandran–Das–S. 2024)

Let \mathcal{F} be a w -bounded θ -intersecting family over $[n]$. Then, there is a bouquet \mathcal{B} in \mathcal{F} such that $|\mathcal{F} \setminus \mathcal{B}| \leq w^3$.

Bounded θ -intersecting families are linear in size!

Theorem (Balachandran–Das–S. 2024)

Let $\theta \in (0, 1) \cap \mathbb{Q}$. Let $w \leq O(n^{1/3})$ be a positive real. There is a constant $C > 0$ such that the following holds: for all sufficiently large n , if \mathcal{F} is a w -bounded θ -intersecting family over $[n]$, then $|\mathcal{F}| \leq Cn$.

Theorem (Balachandran–Das–S. 2024)

Let $\theta = a/b \in (0, 1) \cap \mathbb{Q}$, $\gcd(a, b) = 1$. If \mathcal{F} is a $o(n^{1/3})$ -bounded θ -intersecting family over $[n]$, then $|\mathcal{F}| \leq (C_\theta + o(1))n$, where $C_\theta = \frac{1}{b-a} \sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$. The constant is tight for $\theta \in \{1/3\} \cup [1/2, 1)$.

A linear algebraic reformulation of the original problem

Balachandran–Mathew–Mishra (2019)

Let \mathcal{F} be a θ -intersecting family on $[n]$ of size m .

Let $X_{m \times n}$ be the incidence matrix for \mathcal{F} , so

$$X(A, x) := \begin{cases} 1, & x \in A; \\ -1, & x \notin A. \end{cases}$$

Then,

$$XX^T(A, A) = n,$$

$$XX^T(A, B) = \begin{cases} n - 2|A| + 2(1 - 2\theta)|B|, & \text{if } |A \cap B| = \theta|B|; \\ n - 2|B| + 2(1 - 2\theta)|A|, & \text{if } |A \cap B| = \theta|A|. \end{cases}$$

A linear algebraic reformulation of the original problem

Consider

$$M = \frac{1}{2}(nJ - XX^T).$$

- ▶ The diagonal of M is zero.
- ▶ The (A, B) th entry is either $f(|A|, |B|)$ or $f(|B|, |A|)$, where $f: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ is given by $f(x, y) = x + (1 - 2\theta)y$.
- ▶ If $\text{rank}(M) \geq cm$, then $|\mathcal{F}| \leq \frac{1}{c}(n + 1)$.

A linear algebraic reformulation of the original problem

Let \mathbb{F} be a field, $a_1, \dots, a_n \in \mathbb{F}$, and $f: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ be a function.

Let $\text{Sym}(f; a_1, \dots, a_n)$ denote the collection of all the $n \times n$ symmetric matrices over \mathbb{F} with zero diagonal whose (i, j) th entries are either $f(a_i, a_j)$ or $f(a_j, a_i)$, for all $i < j$.

Question.

Is there an absolute constant $c > 0$ such that $\text{rank}(M) \geq cn$ for all $M \in \text{Sym}(f; \mathbf{a}_1, \dots, \mathbf{a}_n)$?

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Answer.

No!

A matrix of rank at most 3

Consider the matrix

$$M_n = \begin{bmatrix} 0 & 1 & 4 & \cdots & (n-1)^2 \\ 1 & 0 & 1 & \cdots & (n-2)^2 \\ 4 & 1 & 0 & \cdots & (n-3)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)^2 & (n-2)^2 & (n-3)^2 & \cdots & 0 \end{bmatrix},$$

whose (i, j) th entry is $(i - j)^2$. Then $\text{rank}(M_n) \leq 3$ for all $n \geq 3$, since it is the sum of three rank-one matrices.

Good pairs (f, a)

Let \mathbb{F} be a field, a be some sequence in \mathbb{F} , and $f: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ be a function.

1. We say that $(f; a)$ is a *good pair* (over \mathbb{F}) if $f(a_i, a_i) \neq 0$ for all i .

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1. We say that $(f; a)$ is a *good pair* (over \mathbb{F}) if $f(a_i, a_i) \neq 0$ for all i .
2. For $\theta \in \mathbb{F}$, we write $f = f_\theta$ if $f(x, y) = x + (1 - 2\theta)y$ for all $x, y \in \mathbb{F}$.
3. By a_n we mean the tuple (a_1, \dots, a_n) .

Question (refined).

Let $(f; a)$ be a good pair over \mathbb{F} .

Is there an absolute constant $c > 0$ such that $\text{rank}(M) \geq cn$ for all $M \in \text{Sym}(f; a_1, \dots, a_n)$?

Question (refined).

Let $(f; a)$ be a good pair over \mathbb{F} .

Is there an absolute constant $c > 0$ such that $\text{rank}(M) \geq cn$ for all $M \in \text{Sym}(f; a_1, \dots, a_n)$?

Answer.

Yes, with high probability!

A connection with tournaments

A **tournament** over $[n]$ is an orientation of the edges of the complete graph K_n .

Given $f: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ and a sequence a in \mathbb{F} , for a tournament T over $[n]$, define the associated matrix $M_T \in \text{Sym}(f; a_n)$ by

$$M_T(i, j) = \begin{cases} 0, & i = j; \\ f(a_i, a_j), & i < j, i \rightarrow j \text{ in } T; \\ f(a_j, a_i), & i < j, j \rightarrow i \text{ in } T. \end{cases}$$

Conversely, given a matrix $M \in \text{Sym}(f; a_n)$, we can define a tournament over $[n]$ (in possibly more than one way).

Matrices associated to *random* tournaments have high rank

Theorem (Balachandran–Bhattacharya–S. 2023)

Let $\text{char}(\mathbb{F}) \neq 2$, and let \mathbf{a} be a sequence in \mathbb{F}^* . For $\theta \in (0, 1)$, consider the family $\text{Sym}(f_\theta; \mathbf{a}_n)$. Then, for a uniformly random tournament T over $[n]$, with high probability we have

$$\text{rank}(M_T) \geq \left(\frac{1}{2} - o(1) \right) n.$$

Matrices associated to *transitive* tournaments have almost full rank

Assume \mathbb{F} has arbitrary characteristic, and $\theta = 1/2$.

Theorem (Balachandran–Bhattacharya–S. 2024)

Let T_n be the transitive tournament on $[n]$ with the natural orientation. For all $n \geq 2$, $\det(M_{T_n}(a))$ satisfies

$$\det(M_{T_n}(a)) = -a_{n-1}^2 \det(M_{T_{n-2}}(a)) - 2a_{n-1} \det(M_{T_{n-1}}(a)).$$

Corollary (Balachandran–Bhattacharya–S. 2024)

If T is a transitive tournament on $[n]$, then

$$\text{rank}(M_T(a)) \geq n - 1.$$

Question.

Let $(f; a)$ be a good pair over \mathbb{F} .

How low can the rank of a matrix $M \in \text{Sym}(f; a_n)$ be?

Low-rank matrices associated to tournaments

- ▶ The best-known low-rank matrices are those arising from the maximal θ -intersecting families \mathcal{F}_S and \mathcal{F}_H , and these have rank $\frac{2n}{3}$.
- ▶ Let a take only two distinct values, say α and β .
- ▶ Let $(\alpha, \dots, \alpha, \beta, \dots, \beta)$ be denoted by $(\alpha^{(m)}, \beta^{(n)})$.

Bipartite graphs associated to matrices in $\text{Sym}(f; \alpha^{(m)}, \beta^{(n)})$

Note that $\text{Sym}(f; \alpha^{(m)}, \beta^{(n)})$ has the following block matrix form:

$$\begin{bmatrix} f(\alpha, \alpha)(J_m - I_m) & C \\ C^T & f(\beta, \beta)(J_n - I_n) \end{bmatrix}.$$

Here, each entry of C is either $f(\alpha, \beta)$ or $f(\beta, \alpha)$.

Let G_M be the bipartite graph defined by the $f(\alpha, \beta)$ entries.

Low-rank matrices M iff high-multiplicity eigenvalues of G_M

Theorem (Balachandran–S. 2024)

Let $M \in \text{Sym}(f; \alpha^{(m)}, \beta^{(n)})$, where $f(\alpha, \alpha)$, $f(\beta, \beta)$, and $f(\alpha, \beta) - f(\beta, \alpha)$ are all nonzero. Let $\mu(f; \alpha, \beta) \in \mathbb{C}$ such that

$$(\mu(f; \alpha, \beta))^2 = \frac{f(\alpha, \alpha)f(\beta, \beta)}{(f(\alpha, \beta) - f(\beta, \alpha))^2}.$$

If ν is the multiplicity of $\mu(f; \alpha, \beta)$ as an eigenvalue of G_M , then

$$m + n - 2 - \nu \leq \text{rank}(M) \leq m + n + 2 - \nu.$$

In particular, if $f(\alpha, \alpha)f(\beta, \beta) < 0$, then $\text{rank}(M) \geq m + n - 2$.

Low-rank *real* matrices in $\text{Sym}(f_\theta; \alpha^{(m)}, \beta^n)$

Theorem (Balachandran–S. 2024)

Let $\theta \in (0, 1)$. Let $\beta \in \mathbb{R}$ be a root of the quadratic equation $x^2 - (2 + (\theta^{-1} - 1)^2)x + 1 = 0$. For every $n \in \mathbb{N}$, there is a matrix $M \in \text{Sym}(f_\theta; 1^{(n)}, \beta^{(n)})$ with $\text{rank}(M) \leq n + 3$.

These matrices are constructed from the complete bipartite graph $K_{n,n}$ minus a perfect matching.

Low-rank *rational* matrices in $\text{Sym}(f_{1/2}; \alpha^{(m)}, \beta^{(n)})$

Theorem (Balachandran–S. 2024)

For each $\varepsilon > 0$, there exists $c_\varepsilon \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ and $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{Q}$ such that there is a sequence of matrices $M_n \in \text{Sym}(f_{1/2}; \alpha_\varepsilon^{(n)}, \beta_\varepsilon^{(n)})$ for which $\text{rank}(M_n) \leq c_\varepsilon n + O(1)$.

These matrices are constructed from **Hadamard designs**.

Ruling out candidates for low-rank matrices

Proposition (Balachandran–S. 2024)

Let Δ be a symmetric 2 -(v, k, λ) design with $k - \lambda = p^m$ for some prime p and integer $m \geq 1$. Consider $M_\Delta \in \text{Sym}(f_{1/2}; \alpha^{(v)}, \beta^{(v)})$. If $\text{rank}(M_\Delta) \leq v + 3$, then $p^m = 2$.

This rules out most of the known examples of symmetric designs as viable candidates for inducing matrices of low rank.

For instance, none of Families 1, 7, 8, 9, 13, or 14 from the *Handbook of Combinatorial Designs* are viable, except for the Fano plane $PG(2, 2)$ from Family 1.

The Fano family

If we demand $\alpha : \beta :: 1 : 2$, then the Hadamard construction is longer helpful.

In fact, the Fano plane is the only symmetric design that produces matrices of low-rank and with nonzero entries in the proportion $1 : 2$ (Gordon Royle, MathOverflow).

This construction gives matrices $M_n \in \text{Sym}(f_{1/2}; 1^{(7n)}, 2^{(7n)})$ of rank at most $\frac{4}{7}n$.

Furthermore, for $n = 1$, there is an associated $\frac{1}{2}$ -intersecting family $\mathcal{F}_{\text{Fano}}$ over $[8]$ of size 14:

$$\mathcal{F}_{\text{Fano}} = \mathcal{F}_s \cup \{1357, 1368, 1458, 1467\}.$$

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