Tight Bounds on the Carathéodory and Exchange Numbers for $\triangle$-Convexity Spaces[1]

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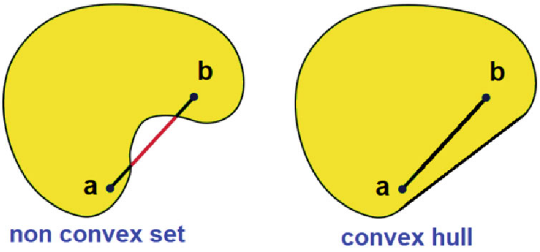
[1] Based on joint work with Vishnu Kumar (IIT Jodhpur)

Convexity in geometry

Consider the d-dimensional Euclidean space, $\mathbb{R}^d$.

A set $S \subseteq \mathbb{R}^d$ is *convex* if S contains the line segment between any two points in S.

The *convex hull* of a set $S \subseteq \mathbb{R}^d$, $\mathrm{Hull}(S)$, is the smallest convex set in $\mathbb{R}^d$ containing S.



Convexity in geometry

- $\emptyset$ and $\mathbb{R}^d$ are convex.
- If $\mathcal{F}$ is any collection of convex sets in $\mathbb{R}^d$, then so is $\bigcap \mathcal{F}$.
- For a convex set S, let $\mathrm{Ext}(S)$ be the set of its *extreme points*, i.e., those points $x \in S$ such that $x \notin \mathrm{Hull}(S \setminus \{x\})$.

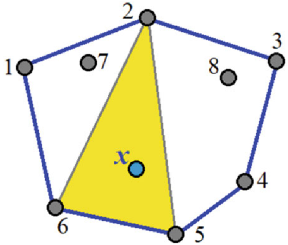
Theorem (Minkowski–Steinitz–Krein–Milman)

Let $S \subseteq \mathbb{R}^d$ be a convex set that is closed and bounded. Then, $S = \mathrm{Hull}(\mathrm{Ext}(S))$.

A classic theorem of convex geometry

Theorem (Carathéodory 1911)

Given $S \subseteq \mathbb{R}^d$, if $x \in \mathrm{Hull}(S)$, then there exists a subset $C \subseteq S$ such that $x \in \mathrm{Hull}(C)$ and $|C| \le d+1$.



Abstract convexity spaces

A *(finite) convexity space* is a finite set X along with a collection $\mathcal{C}$ of subsets of X that satisfies:

1. $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$;
2. if $A, B \in \mathcal{C}$, then $A \cap B \in \mathcal{C}$.

The members of $\mathcal{C}$ are called *convex* sets.

For any subset S of X, its *convex hull*, $\mathrm{Hull}(S)$, is the smallest convex set containing S.

Examples

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3. Let $X = \mathbb{N}$ and $\mathcal{C} = \{\emptyset, \mathbb{N}\} \cup \{[k] : k \in \mathbb{N}\}$.

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3. Let $X = \mathbb{N}$ and $\mathcal{C} = \{\emptyset, \mathbb{N}\} \cup \{[k] : k \in \mathbb{N}\}$.
4. Define the set of *extreme points* of a convex set $C \in \mathcal{C}$ as

$$\mathrm{Ext}(C) = \{x \in C : x \notin \mathrm{Hull}(C \setminus \{x\})\}.$$

A convexity space $(X, \mathcal{C})$ is called a *convex geometry* if it satisfies the Minkowski–Steinitz–Krein–Milman condition: for every $C \in \mathcal{C}$, $C = \mathrm{Hull}(\mathrm{Ext}(C))$.

For example, if $X = \{a, b\}$ and $\mathcal{C} = \{\emptyset, \{a, b\}\}$, then this is *not* a convex geometry.

The Carathéodory number of an abstract convexity space

Let $(X, \mathcal{C})$ be a convexity space. A set $S \subseteq X$ is *Carathéodory independent* if

$$\mathrm{Hull}(S) \setminus \bigcup_{x \in S} \mathrm{Hull}(S \setminus \{x\}) \neq \emptyset.$$

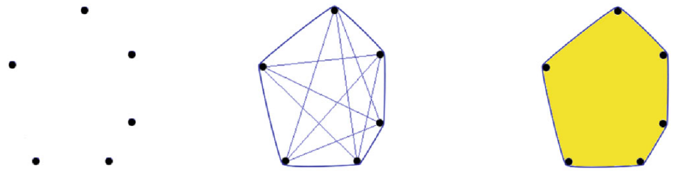
The *Carathéodory number* of X, $c(X)$, is the size of a largest Carathéodory independent subset of X.

Equivalently, the Carathéodory number of X is the smallest integer $r \geq 0$ such that for every subset $S \subseteq X$ and $x \in \mathrm{Hull}(S)$, there exists $F \subseteq S$ where $|F| \leq r$ and $x \in \mathrm{Hull}(F)$.

Carathéodory's theorem says that $c(\mathbb{R}^d) \leq d + 1$.

Intervals in $\mathbb{R}^d$

In $\mathbb{R}^d$, the convex hull $\mathrm{Hull}(S)$ can be generated by successively taking the union of all possible line segments between any two points in the collection.



Intervals in abstract convexity spaces

Given a set X, a function $I\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an *interval function* if it satisfies the following properties:

- Extensivity: $S \subseteq I(S)$ for all $S \in \mathcal{P}(X)$;
- Monotonicity: $S \subseteq S' \Rightarrow I(S) \subseteq I(S')$ for all $S, S' \in \mathcal{P}(X)$;
- Normalization: $I(\emptyset) = \emptyset$.

The collection $\mathcal{C} = \{S \subseteq X : I(S) = S\}$ defines a convexity space over X which is *induced* by the interval function.

Convexity in graphs

For a finite, simple graph $G = (V, E)$, a convexity space over the vertex set V is also called a *graph convexity*. The most commonly studied notions of graph convexity are all defined in terms of interval functions.

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3. P_3 convexity (Centeno–Dourado–Szwarcfiter 2009):
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4. Examples arising from *threshold functions* $\tau : V(G) \to \mathbb{N}$.
   Define the τ-interval $I_\tau(S) = S \cup \{x \in V(G) : |N(x) \cap S| \geq \tau(x)\}$.
   The collection $\mathcal{C}_\tau = \{S \subseteq V(G) : I_\tau(S) = S\}$ defines a graph convexity if and only if $\tau(v) > 0$ for all $v \in V(G)$.

$\triangle$-convexity (Mulder 2008)

Given a finite simple graph $G = (V, E)$, define the interval function $I\colon \mathcal{P}(V) \to \mathcal{P}(V)$ by

$$I(S) = S \cup \{x \in V(G) : x \text{ forms a triangle with two vertices in } S\}.$$

The $\triangle$-*convexity space* over G is the graph convexity induced by this interval function:

$$\mathcal{C} = \{S \subseteq V(G) : \text{ if } x \text{ forms a triangle with } y, z \in S, \text{ then } x \in S\}.$$

The Carathéodory number, $c_\triangle(G)$, is the size of a largest Carathéodory independent subset $S \subseteq V(G)$, i.e., one which satisfies:

$$\mathrm{Hull}(S) \setminus \bigcup_{x \in S} \mathrm{Hull}(S \setminus \{x\}) \neq \emptyset.$$

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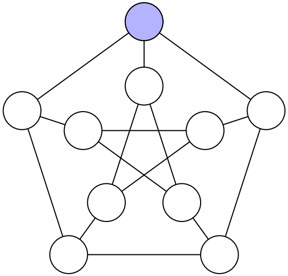
  (Exercise.)

Examples

- Triangle-free graphs:
  $c_{\triangle}(G) = 1$



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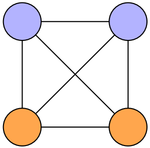
  $c_{\triangle}(K_n) = 2$ for all $n \geq 3$.



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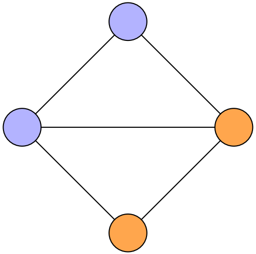
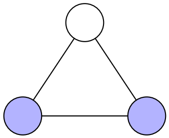
- $c_\triangle(K_4^-) = 2$.

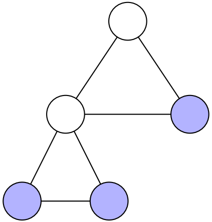


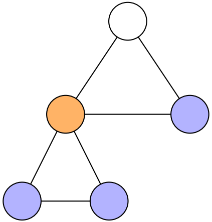
Examples... that are extremal?

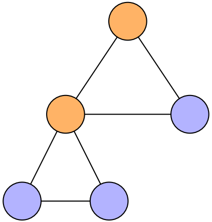
Theorem (Anand–Anil–Changat–Narasimha-Shenoi–Ramla 2025)

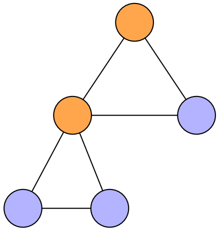
For any finite simple graph G, $c_\triangle(G) \leq t(G) + 1$, where $t(G)$ is the number of triangles in G.

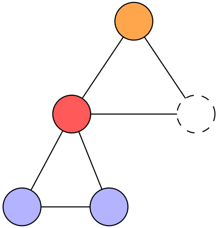


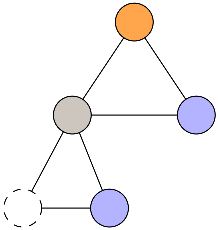


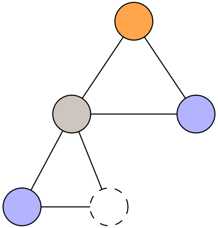


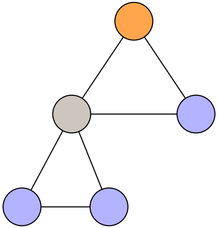


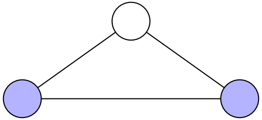


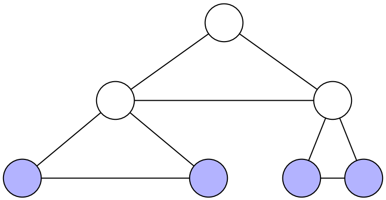


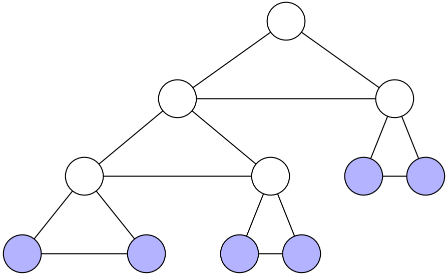


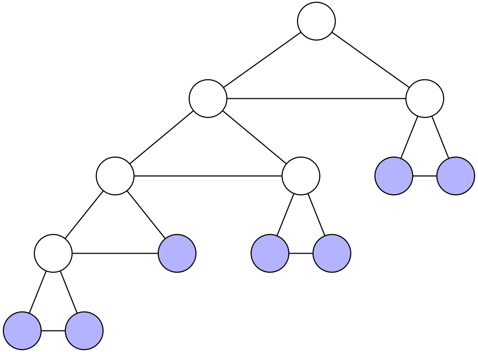


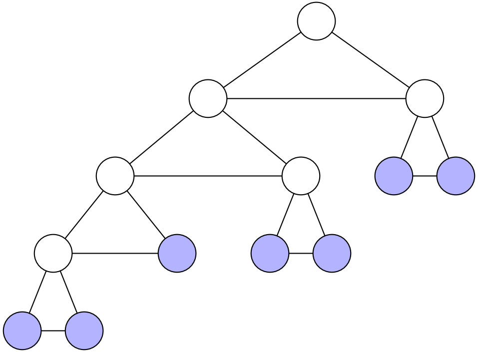


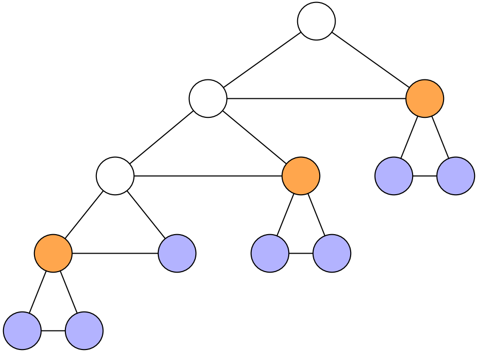


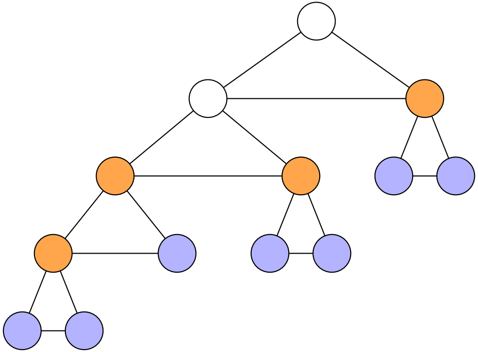


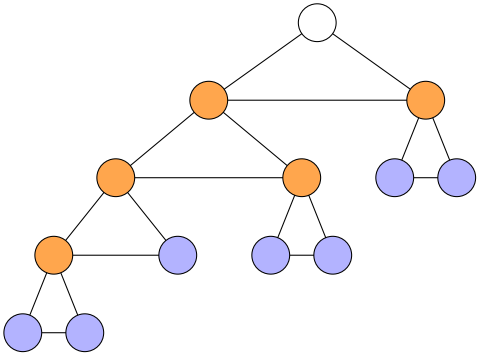


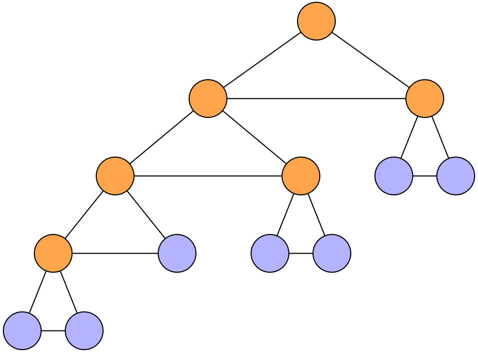


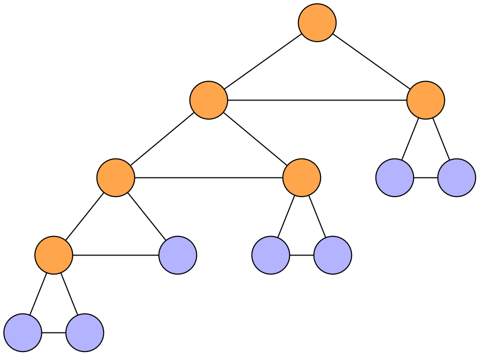


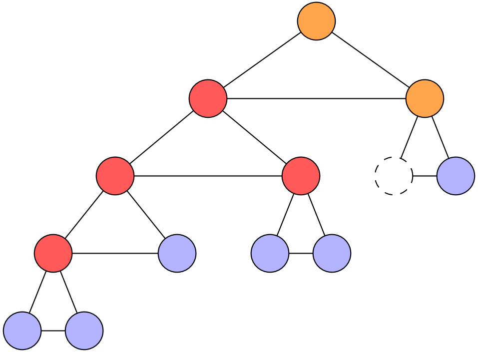


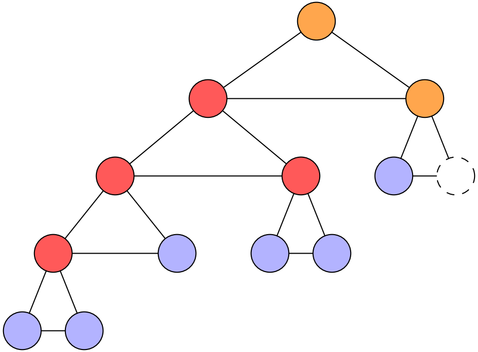


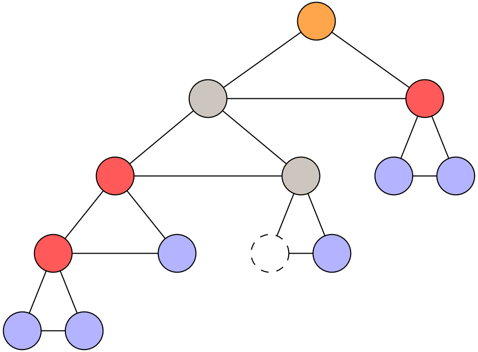


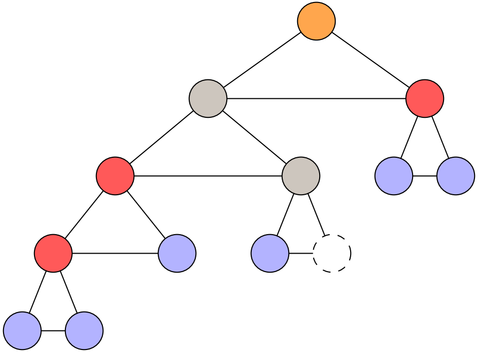


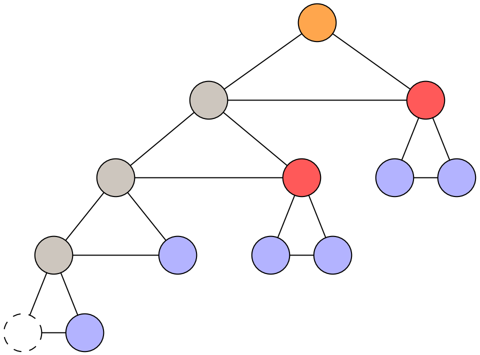


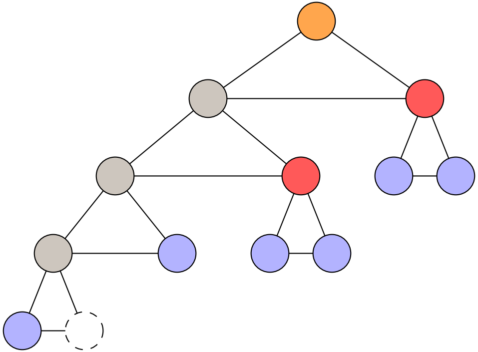


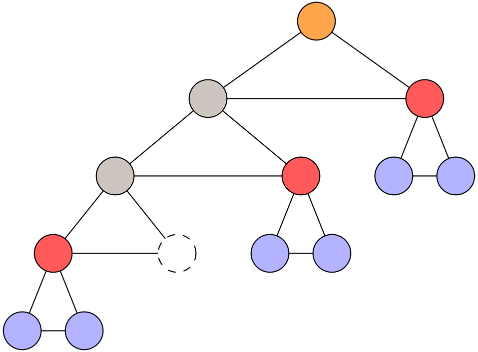


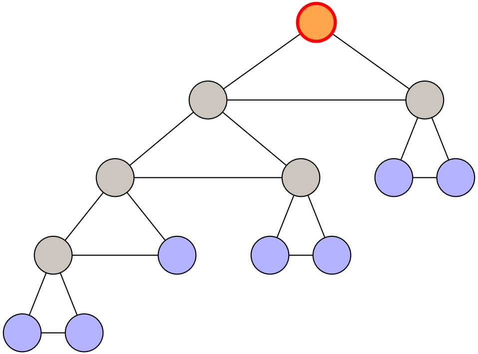












An upper bound on $c_\triangle(G)$

Theorem (Anand–Anil–Changat–Narasimha-Shenoi–Ramla 2025)

For any finite simple graph G, $c_\triangle(G) \leq t(G) + 1$, where $t(G)$ is the number of triangles in G.

Proof outline.

Suppose S is Carathéodory independent with $|S| \geq t(G) + 2$. Then, by the pigeonhole principle, there are at least two triangles containing exactly two vertices of S. Say $T_1, \ldots, T_r$ $(r \geq 2)$ are the triangles with $u_i, v_i \in V(T_i) \cap S$ for each $i \in [r]$. Then,

$$\mathrm{Hull}(S) = \bigcup_{i=1}^{r} \left(\mathrm{Hull}(S \setminus \{u_i\}) \cup \mathrm{Hull}(S \setminus \{v_i\}) \right),$$

so S is Carathéodory dependent. $\square$

We found that it is troublesome to rigorously justify the displayed equation, though it seems intuitively clear, so we provide an alternative proof for the above result.

The main idea

Our proof is by induction on $t(G)$. Let $S \subseteq V(G)$ be a Carathéodory independent set.
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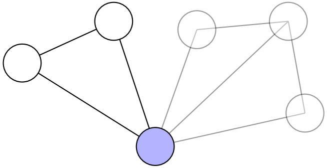
If we found one triangle per vertex, then we would have $|S| \leq t(G)$.
This is clearly impossible, but let's see how far this idea can be pushed.

The main idea

- Isolated vertex: $|\mathcal{T}_H| \geq 1 = |V(H)|$.



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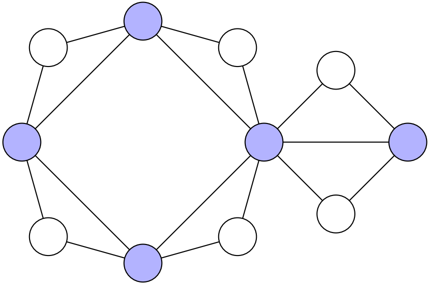
- Component with a cycle: $|\mathcal{T}_H| \geq |E(H)| \geq |V(H)|$.



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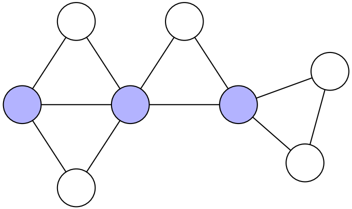
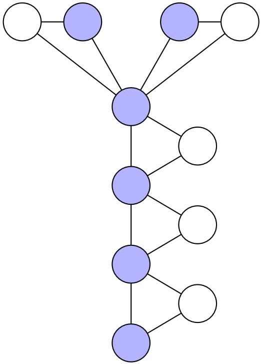
- Tree with a spare: still $|\mathcal{T}_H| \geq |V(H)|$.



The main idea

- Isolated vertex: $|\mathcal{T}_H| \geq 1 = |V(H)|$.
- Component with a cycle: $|\mathcal{T}_H| \geq |E(H)| \geq |V(H)|$.
- Tree with a spare: still $|\mathcal{T}_H| \geq |V(H)|$.
- Exact tree: $|\mathcal{T}_H| = |V(H)| - 1$.



The main idea

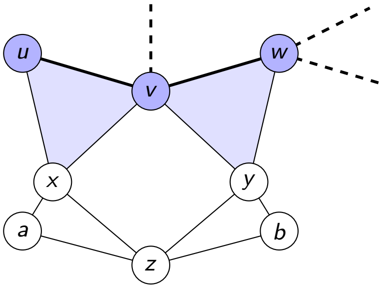
Let $\mathcal{E}$ be the set of exact tree components of $G[S]$. We have

$$\begin{aligned} |S| = \sum_{H} |V(H)| &= \sum_{H \notin \mathcal{E}} |V(H)| + \sum_{H \in \mathcal{E}} |V(H)| \\ &\leq \sum_{H \notin \mathcal{E}} |\mathcal{T}_H| + \sum_{H \in \mathcal{E}} (|\mathcal{T}_H| + 1) \\ &= \sum_{H} |\mathcal{T}_H| + |\mathcal{E}| \\ &\leq t(G) + |\mathcal{E}|. \end{aligned}$$

Thus, if $|\mathcal{E}| \leq 1$, we are done.

Finding spare triangles

- Assume that for each $H \in \mathcal{E}$ and each leaf u of H, there is some path uvw in H for which xvy is part of a 4-cycle, where x and y are the apices of triangles based at uv and vw, respectively.



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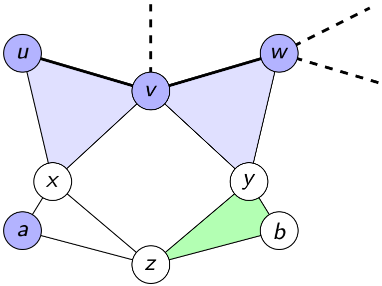
- Then, either xza or yzb or both are spare triangles.



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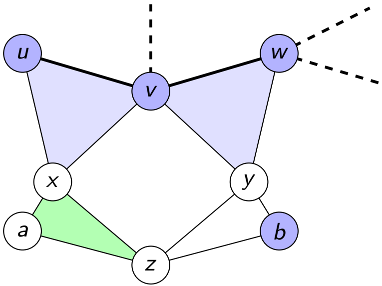

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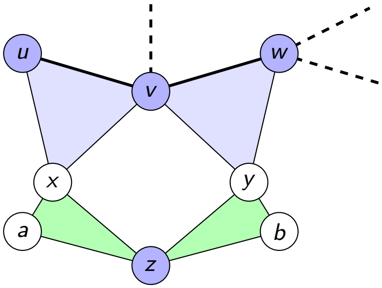

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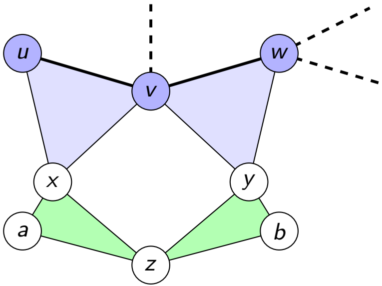

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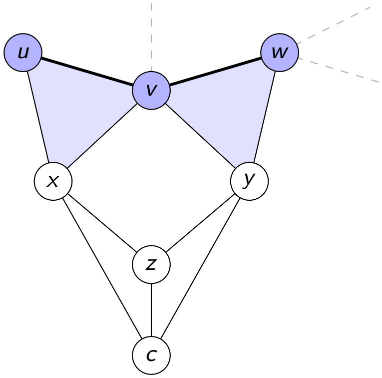
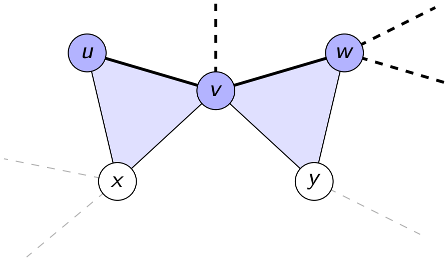
- The argument also works in the case $a = b$.



The inductive step

- Assume that there is a leaf u of some $H \in \mathcal{E}$ such that for every path uvw in H, the path xvy is not part of a 4-cycle, where x and y are the apices of triangles based at uv and vw, respectively.



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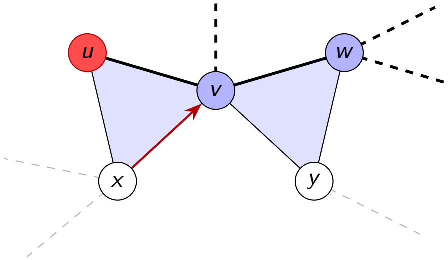
- Then, delete u and contract xv to v^*.



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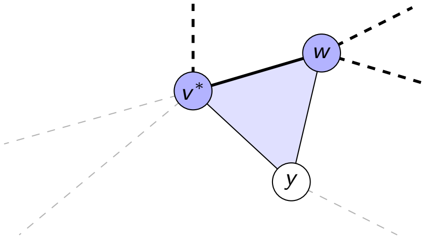
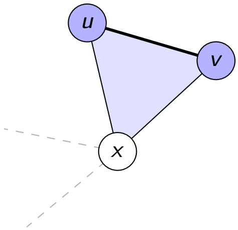


The "edge" case

- Suppose $H \in \mathcal{E}$ has no path of length 2. Then H consists of a single edge uv based at a unique triangle with apex x.



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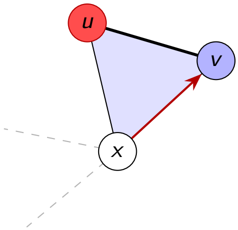

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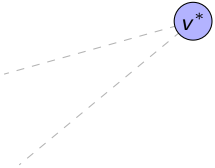

The extremal examples

The proof shows that unless we are in the "edge" case, we get $c_\triangle(G) \leq t(G)$.

The "edge" case shows that we can recursively reduce the graph to a single vertex. This process in reverse is exactly the construction based on full binary trees.

Thus, we have an exact characterization of the graphs for which $c_\triangle(G) = t(G) + 1$.

Theorem (Kumar–S. 2026)

Let G be a finite simple connected graph in which every edge lies on a triangle. If $c_\triangle(G) = t(G) + 1$, then G is obtained from a full binary tree with $c_\triangle(G)$ many leaves by adding an edge between every pair of siblings.

Exchange properties in convexity spaces

The notion of an abstract convexity space is quite close to that of a matroid, and exchange properties have been studied for convexity spaces as well. One such notion goes as follows.

A set $S \subseteq X$ is *exchange independent* if $|S| = 1$ or there exists $p \in S$ such that

$$\mathrm{Hull}(S \setminus \{p\}) \setminus \bigcup_{x \in S \setminus \{p\}} \mathrm{Hull}(S \setminus \{x\}) \neq \emptyset.$$

The *exchange number* of X, $e(X)$, is the size of a largest exchange independent subset of X.

Bounds on $e_\triangle(X)$

Theorem (Sierksma 1984)

For any convexity space $(X, \mathcal{C})$, $e(X) \leq c(X) + 1$.

Corollary (Anand–Anil–Changat–Narasimha-Shenoi–Ramla 2025)

For any finite simple graph G, $e_\triangle(G) \leq t(G) + 2$.

Our proof also characterizes the exchange-extremal graphs:

Theorem (Kumar–S. 2026)

Let G *be a finite simple connected graph in which all but at most one edge lies on a triangle. If* $e_\triangle(G) = t(G) + 2$, *then* G *is obtained from a full binary tree with* $e_\triangle(G) - 1$ *many leaves by adding an edge between every pair of siblings, and attaching a leaf at any one vertex.*

$c_\triangle(G)$ and $e_\triangle(G)$ for block graphs G

In a block graph, each $K_{\geq 3}$-block behaves like a single triangle in terms of the sets generated by taking convex hulls in $\triangle$-convexity. Thus,

Theorem (Kumar–S. 2026)

For a block graph G, let H_0 denote a maximal connected block subgraph without any K_2-blocks, and let H_1 denote a maximal connected block subgraph with exactly one K_2-block, which is also a pendant block of H_1. Then, $c_\triangle(G) \leq \max_{H_0}\{b(H_0)\} + 1$ and $e_\triangle(G) \leq \max_{H_0,H_1}\{b(H_0), b(H_1)\} + 1$, where $b(H_i)$ is the number of blocks of H_i.

This reproves (and fixes the hypotheses of) two results of Anand–Anil–Changat–Narasimha-Shenoi–Ramla (2025). Our proof method also characterizes the extremal graphs in these cases.

Concluding remarks

- The statement of the main result, namely $c_\triangle(G) \leq t(G) + 1$, tempts one to look for a proof that produces a natural injection from a Carathéodory independent set of vertices to some set $\mathcal{T}^*$ of triangles of G with an additional element adjoined to it.

  The current proof does not appear to be of this form, though it does help characterize the extremal graphs.

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- The most commonly studied graph convexity spaces are either path convexities, or they arise from some threshold functions. Curiously, $\triangle$-convexity is of neither kind.

- The computational complexity for convexity parameters is of broad interest. Some results have been obtained for $\triangle$-convexity previously by Anand–Anil–Changat–Dourado–Ramla (2020) and Anand–Dourado–Narasimha-Shenoi–Ramla (2022).

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